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Comparing Expectational Stability Criteria in Dynamical Models: a Preparatory Overview

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Abstract

This paper compares the most significant expectational stability criteria that have been used to assess the plausibility of perfect foresight trajectories in forward-looking dynamical systems: determinacy of trajectories, absence of neighbour sunspot trajectories, and convergence of "evolutive" and "eductive" learning processes. It examines, within a set of increasingly complex dynamical models, the robustness of an equivalence principle suggested by the analysis of the simplest classes of those models.

Résumé

On compare dans ce texte les critères de stabilité des anticipations les plus communément utilisés dans les modèles économiques dynamiques à horizon infini : "détermination" de l'équilibre, absence d'équilibres à taches solaires voisins, succès de l'apprentissage ou bien "évolutif" ou bien "divinatoire". On montre que le principe d'équivalence, suggéré par l'examen des modèles les plus simples et convenablement réinterprété, vaut, quoique de façon affaiblie, dans des classes de modèles de complexité croissante.

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Keywords: rational expectations, determinacy, sunspots, learning.

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1 Introduction

Economists have stressed a variety of viewpoints in order to assess the expectational stability of perfect-foresight or rational-expectations equilibria. The coexistence of these various approaches is particularly striking in the case of infinite-horizon dynamical models.

Let us review it briefly.

The local *determinacy* viewpoint stresses equilibria that are locally "isolated" or locally "determinate": if this property does not hold, the strong joint assumptions that agents know the model, know some neighborhood of the outcome and have perfect foresight have no predictive power.

A similar argument suggests that an equilibrium is more likely to occur if it is locally *immune to sunspot* fluctuations: in a well chosen neighborhood of the equilibrium, there is no regular enough stochastic equilibria. Finally, the *learning* approach relaxes the assumption that agents already know the set of rational expectations equilibria, and try to coordinate their behavior on such an equilibrium. In the *"evolutive" learning* approach, one supposes that agents follow plausible rules of thumb: the system reacts in real time and possibly (this is the success criterion) converges toward the sought after equilibrium. In the *"eductive" learning* approach, the expectational plausibility of a candidate equilibrium is evaluated through tests that reflect collective mental processes, and that are successfully passed by Eductively Stable, or Strongly Rational, equilibria.

It turns out that all these viewpoints are less different from one another than they appear, at least in the framework of a simple prototype overlapping generations model (from now OLG) in which Guesnerie (1993) has emphasized an "equivalence theorem". Indeed, in such an OLG model, all the four criteria under scrutiny recommend that the same steady state equilibrium be selected as "expectationally stable".

However, many distinct results about determinacy, sunspots and learning have been obtained in much more complex dynamical systems. It is natural to confront these results to the questions raised by the "equivalence theorem". What are the connections, outside the simple OLG model alluded above, between determinacy, sunspot immunity, "evolutive" and "eductive" learning viewpoints?

In order to answer this question, the present paper proceeds as follows.

First, in Section 2, it restates, within the linearized version of a one-dimensional, one-step forward looking model without memory, the "equivalence theorem".

Then, in Section 3, it turns to the class of one-dimensional, one-step forward looking, memory-one models, and argues that a weaker version of the "equivalence theorem" holds.

In Section 4, the same exercise is repeated for the class of one-dimensional, one-step forward looking, finite memory models.

In Section 5, we shall consider the class of n-dimensional, one-step forward looking, memory-one models.

A brief conclusion is finally offered.

2 A dynamic equivalence result

2.1 The framework

In this section, we shall consider the simplest abstract economic framework in which agents' forecasts influence the actual evolution of the economic system over time. It is a linear one-step forward system in which the state of the economic system at date t ($t \geq 0$) is expressed as a one-dimensional real variable $p(t)$ determined by the common forecast $E\left[p(t+1) \mid I_t\right]$ of this variable in the next period. The corresponding temporary equilibrium relation is written:

$$\gamma E\left[p(t+1) \mid I_t\right] + p(t) = 0. \tag{1}$$

For convenience, $p(t)$ is to be thought of as the price of some commodity at t. In (1), the mean operator E stands for the agents' forecast, I_t is the information available at t, and γ ($\gamma \neq 0$) is a real parameter that measures the sensitivity of the actual state of the economy to agents' forecasts.

In order to close the model, one must now specify how forecasts are formed. According to the perfect foresight hypothesis, $E\left[p(t+1) \mid I_t\right]$ is equal to the actual price $p(t+1)$ at date $(t+1)$. Under this hypothesis, the dynamics (1) becomes

$$\gamma p(t+1) + p(t) = 0. \tag{2}$$

A perfect foresight equilibrium is a sequence of prices $(p(t), t \geq 0)$ which satisfies (2) in each period. The analysis used to characterize such sequences is straightforward in the class of models (1). In fact, with respect to laws of motion of the form

$$p(t) = \sum_{l=1}^{L} \beta_l p(t-l), \tag{3}$$

it is easy to verify that at most one lagged price is relevant in equilibrium, so that we can focus on the case where $L = 1$ in (3). If agents a priori believe that $p(t) = \beta_1 p(t-1)$, then $E\left[p(t+1) \mid I_t\right] = \beta_1^2 p(t-1)$, and the actual current price $p(t)$ in (1) is equal to $-\gamma\beta_1^2 p(t-1)$. Perfect foresight obtains if and only if the a priori guess about the law of motion coïncides in every period with the actual law, for any possible past history of the price, summarized here by $p(t-1)$. Indeed, it must be the case that $\beta_1 = -\gamma\beta_1^2$.

One can consequently distinguish two types of equilibria within the class of models under consideration.

The first type obtains if $\beta_1 = 0$. Then, the current price is not linked to the past in equilibrium; actually, it remains constant over time. Provided that the regularity assumption $\gamma \neq -1$ is satisfied, the steady state sequence $(p(t) = 0, t \geq 0)$ is the only solution of this type.

In the second type of equilibria, the price evolves over time according to the first order law $p(t) = -p(t-1)/\gamma$. Since there is no restriction on the initial agents' forecasts, there are infinitely many equilibria of this kind. Each one is associated with an arbitrary forecast $E[p(1) \mid I_0] = p(1)$ formed at $t = 0$, or equivalently with an initial price $p(0)$ for the trajectory.

All this may look like a useless detour for a simple conclusion, but the exercise allows us to introduce a terminology reminiscent of that used by McCallum (1983). A solution along which the current price depends on a minimal number of lagged prices is said to be of minimal order (hereafter an MO solution). The solution is a bubble otherwise. Thus, the steady state sequence is the only MO solution to (2), whereas the first order solutions are bubbles.[1]

2.2 Expectational plausibility of the steady state

We introduce now the criteria evoked in the introduction in a formal way. These criteria provide different assessments about the likelihood that agents collectively succeed to coordinate on some self-fulfilling forecast. Here, we focus attention on the steady state, and the case of bubbles will be examined in Appendix 1.

The first criterion reflects the idea that selecting a locally determinate equilibrium is easier. Indeed, provided that agents restrict their attention to the set of perfect foresight equilibria, a locally unique equilibrium can be thought of as a focal point for the process of beliefs coordination.

Formally, an equilibrium $(p(t), t \geq 0)$ is *determinate*, in a terminology advocated for example by Woodford (1984), if there exists no other equilibrium $(p'(t), t \geq 0)$ such that $|p(t) - p'(t)| < \varepsilon$ in each period, where $\varepsilon > 0$ is any arbitrarily small real number. The equilibrium is *indeterminate* otherwise. If $|\gamma| < 1$, any perfect foresight sequence $(p(t), t \geq 0)$ with $p(0) \neq 0$ is pulled away the vicinity of the steady state. If, on the contrary, $|\gamma| > 1$, there are infinitely many first order solutions $(p(t), t \geq 0)$ such that $p(t)$ remains arbitrarily close to 0 in each period. Hence, the steady state is determinate if $|\gamma| < 1$, and it is indeterminate if $|\gamma| > 1$.

The second criterion assumes that agents focus attention on the whole set of rational expectations equilibria, which includes not only the perfect foresight equilibria, but also the stochastic equilibrium trajectories along which *stationary sunspots* matter. According to this criterion, an equilibrium is more likely to obtain whenever it is immune to sunspots.

[1] An MO-solution corresponds to a MSV-solution in Evans and Honkapohja's (2001) terminology. The McCallum's (1983) MSV-solution (Minimum State Variable) is actually a particular MO-solution. In the present model, in which there is an unique MO-solution, both concepts coincide.

Formally, assume that agents observe a 2-state Markovian sunspot process defined by a 2×2 Markov matrix $\mathbf{\Pi}$, with $\pi(s, s')$ standing for the probability that sunspot event be s' tomorrow if it is s today ($s, s' = 1, 2$). Assume also that they believe that the price is perfectly correlated with sunspots, that is that $p(t)$ should equal $p(s)$ if the sunspot event is s at date t. Their common price forecast $E[p(t+1) \mid s]$ is consequently $\pi(s,1)p(1) + \pi(s,2)p(2)$ if the sunspot event is s at date t. The actual price $p(t)$ is finally obtained by introducing this forecast into the temporary equilibrium map (1). In a sunspot equilibrium, beliefs must be self-fulfilling:

$$p(s) = -\gamma\left[\pi(s,1)p(1) + \pi(s,2)p(2)\right] \tag{4}$$

whatever s is ($s = 1, 2$). Moreover, it must be the case that $p(1) \neq p(2)$ for the price to be subject to endogenous stochastic fluctuations in equilibrium.

In this model, and in line with more general results of, for example, Chiappori and Guesnerie (1989), reprinted in Guesnerie (2001), it is straightforward to show that sunspot equilibria exist if and only if the steady state is indeterminate. One can also refer, both for a proof and an intuition of the existence property to the so-called invariant set argument (see Guesnerie and Woodford (1992)).[2]

Unlike the previous criteria, the next one does not rule out beliefs that would not a priori fit the rational expectations hypothesis. In an *"evolutive" learning* process, agents make forecasts errors and revise their falsified beliefs in the light of these errors. A learnable equilibrium is then seen as more expectationally plausible than an equilibrium that would be unstable in such learning dynamics. As an example, under the simplistic myopic learning rule, agents merely expect the next price to equal the last price they have observed:

$$E\left[p(t+1) \mid I_t\right] = p(t-1). \tag{5}$$

With this forecast rule, the current price $p(t)$ at date t, obtained from (1), is

$$p(t) = -\gamma p(t-1). \tag{6}$$

Thus, agents eventually learn the steady state if and only if $|\gamma| < 1$, or equivalently the steady state equilibrium is determinate in (2).[3]

We now come to the *"eductive" learning* process. Let us refer first to the concept of Iterative Expectational Stability (hereafter IE-stability). The starting point of the learning process under consideration consists to assume that agents have a guess $p(\tau)$ about the steady state at some step τ ($\tau \geq 0$) of a mental reasoning. If they

[2] If $|\gamma| > 1$, then $[p(1), p(2)]$ is an invariant set in the sense that every stochastic belief regarding the next price whose support lies in $[p(1), p(2)]$ triggers a current state in this set.

[3] The relationship between determinacy and stability in some dynamics with learning belongs to the folklore of the overlapping generations model (an early reference is Stiglitz (1973)). The intuition hinges on the fact that the dynamics with perfect foresight and the learning dynamics are time mirror, at least for myopic learning processes.

all know the structure of the economy, summarized by the temporary equilibrium relation (1), then they all know that if they expect the next price to be $p(\tau)$, it is actually $-\gamma p(\tau)$ today. This should urge agents to revise their guess about the steady state as $p(\tau+1) = -\gamma p(\tau)$.[4] Thus, for τ large enough, they eventually convince that $p(\tau) = 0$ if and only if $|\gamma| < 1$, in which case the steady state is determinate.

It is worth recalling here that IE-stability is equivalent to eductive stability of the steady state whenever "eductive" stability is explicitly defined in a setting in which individual and collective reactions to expectations are weakly compatible. When the alluded conditions do not hold, IE-stability becomes a necessary condition of eductive stability.[5]

The analysis of this section is summarized by the following proposition.

Proposition 1. Equivalence principle in one-step forward, one-dimensional linear systems.

Consider a one-step forward looking model (1) *with* $\gamma \neq 0$. *Then, the following three statements are equivalent:*

1. The steady state is determinate.

2. The steady state is immune to (stationary) sunspots.

3. For a given "reasonable" learning rule, the steady state in asymptotically stable.

4. The steady state is IE-stable.

Statement 3 of Proposition 1 follows from Guesnerie (1993). It is in fact much stronger than what consideration of the myopic learning rule, introduced here would allow to be inferred. Actually, "reasonable" refers to a wider class of learning processes examined in Guesnerie and Woodford (1991): adaptive learning rules that detect cycles of order 2. Statement 3 is particularly strong; it says that if one takes any given reasonable learning rule and fixes it, its asymptotic stability obtains if and only if the other statements are true.

The present paper will demonstrate that the equivalence result given in Proposition 1 above, though it is established in a simple class of models, has broader implications.

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[4]Note that the IE dynamics is here the same as the myopic learning dynamics.

[5]To see this point most clearly, we may assume that all traders believe at date t that the state value of the price will be in some interval of possible prices $P^T = [P^T_{\inf}, P^T_{\sup}]$, with $P^T_{\inf} < 0 < P^T_{\sup}$, at any period $T > t$. Provided that this fact is Common Knowledge, they can infer that $E[p(T) \mid I_{T-1}]$ will be in P^T at date $(t-1)$. Now, if they all know (1), and *if their reaction functions to expectations have the same monotonicity properties*, they infer that the state value of the price $p(T-1)$ in period $T-1 > t$ is in $P^{T-1} = [P^{T-1}_{\inf}, P^{T-1}_{\sup}]$, with $P^{T-1}_{\inf} = -\gamma P^T_{\inf}$, and $P^{T-1}_{\sup} = -\gamma P^T_{\sup}$ (note that we have set $\gamma < 0$ for convenience). If T is arbitrarily large, then all the traders know in the current period that $E[p(t+1) \mid I_t] = 0$ if and only if $|\gamma| < 1$, in which case the price is equal to its steady state value 0 from t onwards.

3 The effect of memory

3.1 The framework

The class of models examined in the previous section has no predetermined variables in the temporary equilibrium map. This is restrictive since the past matter in many economic situations: it is the case in widely studied models with capital accumulation in which the current stock depends both on the future (through expected future demand) and on the past (through the accumulated capital stock).

Let us accordingly consider the following temporary equilibrium map:

$$\gamma E\left[k(t+1) \mid I_t\right] + k(t) + \delta k(t-1) = 0, \tag{7}$$

where γ and δ are real parameters ($\gamma, \delta \neq 0$). Such dynamics obtain from linearized versions of overlapping generations models with production, at least for particular technologies (Reichlin (1986)), or infinite horizon models with a cash-in-advance constraint (Woodford (1986)).

A perfect foresight equilibrium is now a sequence $(k(t), t \geq -1)$ such that

$$\gamma k(t+1) + k(t) + \delta k(t-1) = 0 \tag{8}$$

in any period $t \geq 0$, given the initial condition $k(-1)$. As in the previous section, the entire set of these equilibria can be analyzed by appealing to the method of undetermined coefficients. One can show that it is enough to consider lagged solutions of the form:

$$k(t) = g_1 k(t-1) + g_2 k(t-2), \tag{9}$$

where g_1 and g_2 are two real parameters to be determined. Then, it is easy to show (and left to the reader) that perfect foresight obtains if and only if

$$g_1 = -\left[\gamma\left(g_1^2 + g_2\right) + \delta\right] \tag{10}$$

and

$$g_2 = -\gamma g_1 g_2. \tag{11}$$

Lagged solutions are consequently of two kinds. If $g_2 = 0$, then (11) is satisfied. In this case, in equilibrium, the current stock of capital only relates to the previous one, and g_1 is the solution of the equation $g_1 = -\gamma g_1^2 - \delta$. If the two solutions, denoted λ_1 and λ_2, are complex valued, there is no equilibrium along which the capital stock is linearly related to the previous capital stock. If they are real, then there is either one or two solutions of this form, depending on whether λ_1 and λ_2 are identical or different, respectively.

Subsequently, we shall assume that λ_1 and λ_2 are real (which arises if and only if $1 - \delta\gamma \geq 0$) and that they have different moduli (with $|\lambda_1| < |\lambda_2|$ by definition).

Therefore, in the considered class, given the initial condition $k(-1)$, there are two perfect foresight solutions:

$$k(t) = \lambda_1 k(t-1), \tag{12}$$

and

$$k(t) = \lambda_2 k(t-1). \tag{13}$$

Both laws are defined for any $t \geq 0$.[6] Since, along these solutions, the number of lagged variables that influence the current state is equal to the number of predetermined variable, the set of MO solutions comprises only these two first order equilibria.

If $g_2 \neq 0$ in (11), it must be the case that $g_1 = -1/\gamma$ and $g_2 = -\delta/\gamma$. The evolution is then driven by the law

$$k(t) = -\frac{1}{\gamma}k(t-1) - \frac{\delta}{\gamma}k(t-2), \tag{14}$$

for any $t \geq 1$. In the initial period $t = 0$, there is one degree of freedom in the agents' forecasts, unlike the two laws (12) and (13). It follows that, for a given initial state $k(-1)$, there are infinitely many equilibrium paths of the second order type. They are all bubble solutions.

In the literature, an often privileged object of scrutiny is the steady state sequence $(k(t) = 0, t \geq -1)$ of (8). Such a sequence is a perfect foresight equilibrium if and only the initial state $k(-1)$ equals 0. Of course, the determinacy criteria that will be introduced later might be applied to the steady state as well. Still, we prefer to use a related, but different terminology: the steady state is a sink if $|\lambda_2| < 1$, a saddle if $|\lambda_1| < 1 < |\lambda_2|$, or a source if $|\lambda_1| > 1$.

There is actually a widespread opinion in the literature that the so-called "equivalence theorem" fails in the presence of memory. This seems supported by a number of facts. For instance, as shown by Dàvila (1997), whatever the properties of the steady state are, there do not exist sunspot equilibria with a finite support in the state space, in which agents would base their forecasts on the current sunspot event only (although stochastic fluctuations à la Broze-Szafars (1991) or à la Benhabib-Farmer (1997) do occur). If, however, agents use both the current and the previous sunspot events, then Dàvila (1997) has shown that the fact that the steady state is a sink is a necessary, but not sufficient condition for existence of Markovian sunspot equilibria. Moreover, agents do no longer necessarily learn a determinate steady state, even if they use a standard learning scheme (Grandmont and Laroque (1990), (1991)).[7]

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[6]Note that, as defined, g_1 is a root of the characteristic polynomial associated with (8). This shows that the solutions under scrutiny determine trajectories in some one-dimensional invariant subspace of the dynamical system in $I\!R^2$ describing the evolution over time of the vector $(k(t-1), k(t))$ in the class of models (8)

[7]Grandmont and Laroque (1990) show that, in the class of models under scrutiny in this section,

Still, the conclusion of a failure in the dynamic equivalence principle is based on a superficial understanding of the direction of generalization. As argued by Gauthier (2002) and Desgranges and Gauthier (2003), this principle must first be conveniently reconsidered.

3.2 Determinacy of growth rates

First, for such a reconsideration, determinacy has to be viewed as a property of trajectories (and not, as the literature sometimes suggests, of their limit points). Hence, applying our four criteria to the steady state level of capital would be erroneous in general, that is as soon as $k(-1)$ differs from 0.

The second ingredient of such a reconsideration hinges on a reflection about the notion of proximity of trajectories in the new setting. Recalling that a trajectory $(k(t), t \geq -1)$ is determinate if there is no other trajectory $(k'(t), t \geq -1)$ that is close to it, we have now to delineate an appropriate topology. Yet the choice of the suitable topology, which was unambiguous in the previous section, is now open.

The most natural candidate is the C0 topology, used in the previous section, according to which two different trajectories $(k(t), t \geq -1)$ and $(k'(t), t \geq -1)$ are said to be close whenever $|k(t) - k'(t)| < \varepsilon$, for any $\varepsilon > 0$ arbitrarily small, and any date $t \geq -1$. Then, a MO solution driven by a root λ_i is locally determinate as soon as $|\lambda_i| > 1$. In fact, with such a concept of determinacy, the saddle-path solution, along which $k(t) = \lambda_1 k(t-1)$ when $|\lambda_1| < 1 < |\lambda_2|$, is the only non-explosive solution to be locally determinate in the C0 topology.

In the simple class of models without memory analyzed in the previous section, the unique MO solution was defined by a constant *level* of the state variable, and the dynamic equivalence principle was applied in terms of the C0 topology.

In the context of models (7), with memory, a MO solution is characterized by a constant *growth rate* of the state variable, and not by a constant level of the state variable (if both λ_1 and λ_2 differ from 1). This suggests that determinacy should be applied in terms of growth rates, in which case closedness of two trajectories $(k(t), t \geq -1)$ and $(k'(t), t \geq -1)$ would require that the ratio $k(t)/k(t-1)$ be close to $k'(t)/k'(t-1)$ in each period $t \geq 0$. This is an ingredient of a kind of C1 topology, as advocated by Evans and Guesnerie (2003a). In this topology, two trajectories $(k(t), t \geq -1)$ and $(k'(t), t \geq -1)$ are said to be close whenever both the levels $k(t)$ and $k'(t)$ are close, and the ratios $k(t)/k(t-1)$ and $k'(t)/k'(t-1)$ are close, in any period.

As stressed by Gauthier (2002), the examination of proximity in terms of growth rates leads consideration of the dynamics with perfect foresight (8) in terms of

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if the steady state is locally stable for a learning rule that enables agents to detect cycles of period 2, then it is necessarily a saddle point. The reciprocal does not hold in general, however.

growth rates $g(t) = k(t)/k(t-1)$, or equivalently

$$k(t) = g\,(t)\,k(t-1). \tag{15}$$

If (15) holds for any $k(t-1)$ and any $t \geq 0$, then

$$k(t+1) = g\,(t+1)\,k(t) = g\,(t+1)\,g\,(t)\,k(t-1).$$

It follows that, from the dynamics (8), we have

$$k(t) = -\left[\gamma g\,(t+1)\,g\,(t) + \delta\right] k(t-1). \tag{16}$$

Hence, consistency between (15) and (16) requires that

$$g\,(t) = -\left[\gamma g\,(t+1)\,g\,(t) + \delta\right]. \tag{17}$$

Associated with the initial perfect foresight dynamics, is then a *perfect foresight dynamics of growth rates*. This new dynamics is non-linear, and it has a one-step forward looking structure, without predetermined variable. Namely, in (17) the growth factor $g\,(t)$ is determined at date t by the correct forecast of the next growth factor $g\,(t+1)$.

Let us sum up: with any given perfect foresight trajectory of states $(k(t), t \geq -1)$ is associated by (15) a unique sequence of growth rates satisfying (17). If the trajectory is an MO solution, then the growth rates trajectory $(g\,(t)\,, t \geq 0)$ is a steady growth rate whose value, from (17), equals one perfect foresight root λ_i If it is a bubble solution, then the growth rates induced by (17) fluctuates over time. Reciprocally, given any sequence meeting (17), and given $k(-1)$, there exists a unique perfect foresight trajectory $(k(t), t \geq -1)$ solution to (8), since (8) and (17) are consistent by construction.

This first discussion suggests that as soon as determinacy is under examination, the problem can be reassessed, in terms reminiscent of those of Section 2, when growth factors, rather than levels are considered. We are going to pursue this line of thought and argue that it provides the appropriate way to generalize of the equivalence principle.

3.3 Sunspots on growth rates

Maintaining the focus on growth rates, let us now define a concept of sunspot equilibrium, in the neighborhood of MO solutions. Suppose that agents a priori believe that the growth factor of capital, and not the level of capital, is to be perfectly correlated with sunspots. Namely, if the sunspot event is s at date t, they a priori believe that $g(t) = g(s)$, that is

$$k(t) = g(s)k(t-1). \tag{18}$$

Thus, their common forecast is

$$E\left[k(t+1)\mid I_t\right]=\pi(s,1)g(1)k(t)+\pi(s,2)g(2)k(t),$$

where $\pi(s,1)$ and $\pi(s,2)$ are the transition probabilities used in Section 2 and (4). Given this forecast, the current stock, obtained from (7), expresses as

$$k(t)=-\left[\gamma\left[\pi(s,1)g(1)+\pi(s,2)g(2)\right]g(s)+\delta\right]k(t-1). \tag{19}$$

In a sunspot equilibrium, the a priori belief (18) coincides with (19), whatever $k(t-1)$ is. As shown by Desgranges and Gauthier (2003), this consistency condition is written

$$g(s)=-\left[\gamma\left[\pi(s,1)g(1)+\pi(s,2)g(2)\right]g(s)+\delta\right]. \tag{20}$$

For the growth rate to fluctuate, one must also impose that $g(1)\neq g(2)$ in (20).

Comparing with (17), we observe that (20) can be seen as defining directly a sunspot equilibrium on the growth rate, as soon as the stochastic dynamics of growth rates is extended as $g\left(t\right)=-\gamma E\left[g\left(t+1\right)\mid I_t\right]g\left(t\right)-\delta$. Yet the equivalence of the two definitions of the concept of sunspot equilibrium on growth rates suggested here relies on special assumptions about linearity and certainty equivalence.

3.4 Eductive learning of growth rates

Assume finally that agents are not aware of steady growth factors, and try to discover a MO solution. The discussion of the basic viewpoint of eductive learning would require that some game theoretical flesh be given to the dynamical model under scrutiny, as in Evans and Guesnerie (2003a,b). Here, we shall instead refer to the more informal IE-stability criterion. Let agents a priori believe that the law of motion of the economy is given by

$$k(t)=g(\tau)k(t-1), \tag{21}$$

where $g(\tau)$ denotes the conjectured growth rate at step τ in some mental reasoning process. Given (21), they expect the next stock of capital to be $g(\tau)k(t)$, so that the actual stock is $k(t)=-\delta k(t-1)/(\gamma g(\tau)+1)$. Assume that all the agents understand that the actual growth factor is $-\delta/(\gamma g(\tau)+1)$ when their initial guess is $g(\tau)$, they should revise their guess as

$$g(\tau+1)=-\frac{\delta}{\gamma g(\tau)+1}. \tag{22}$$

This is the IE-stability criterion. By definition, IE-stability obtains whenever the sequence $(g(\tau),\tau\geq 0)$ converges in (22) toward one of its fixed point, a fact that is interpreted as reflecting the success of some mental process of learning.[8] Since this dynamics is the time mirror of the perfect foresight dynamics of growth rate, a fixed

[8]Note that, here as in the previous section, a steady growth rate is locally stable in a myopic learning dynamics bearing on growth rates if and only if it is locally IE-stable in (22). Indeed, let the

point λ_1 or λ_2 is locally IE-stable if and only if it is locally unstable in (17), that is locally determinate.

IE-stability is a necessary condition of eductive stability. As underlined in Evans and Guesnerie (2003a), the hypothetical Common Knowledge of growth rates triggers a mental process that, in successful case, progressively reinforces the initial restriction and converges toward the solution. The mental process takes into account the variety of beliefs associated with the initial restriction: common beliefs with point expectations is then a particular case, and it is intuitively plausible that convergence of the general mental process under consideration implies convergence of the special process under examination when studying IE-stability.

3.5 An equivalence in models with one memory

Following previous discussions and definitions, we are now in a position to make explicit the connections between the different viewpoints introduced.

Proposition 2. Equivalence principle in one-step forward, memory one, one-dimensional linear systems.

Consider a one-step forward looking model (7) *with one lagged predetermined variable, where* $\gamma, \delta \neq 0$. *Assume that both* MO *solutions exist, that is,* λ_1 *and* λ_2 *are real. Assume finally that they have different stability properties, that is,* $|\lambda_1| < |\lambda_2|$. *Then the following four statements are equivalent:*

1. A constant growth rate MO *solution is locally determinate in the perfect foresight growth rate dynamics.*

2. A constant growth rate MO *solution is locally immune to (stationary) sunspots on growth rates.*

3. For any a priori given "reasonable" learning rules bearing on growth rates, a constant growth rate MO *solution is locally asymptotically stable.*

4. A constant growth rate MO *solution is locally IE stable.*

Moreover, there is only one MO *solution satisfying 1 to 4, the one along which the constant growth rate is equal to the perfect foresight root* λ_1 *of smallest modulus.*

If this MO *solution defines a converging trajectory* ($|\lambda_1| < 1$), *then conditions 1 to 4 are equivalent to the fact that this trajectory is determinate in the C1 topology of trajectories.*

Proof. The paper does not normally reproduce existing proofs. However, we give the proof of Statement 1 proposed by Gauthier (2002) in Appendix 2.

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agents' estimate of the growth rate be $g(t)$ at outset of period t. Their forecast about the next state is consequently $g(t)k(t)$, so that the current capital stock is such that $\gamma g(t)k(t)+k(t)+\delta k(t-1) = 0$. In other words, if the estimate of the growth rate at date t is $g(t)$, the actual growth rate is $-\delta/(\gamma g(t)+1)$. The myopic learning rule coincides with (22), after replacing virtual time τ by real time t.

Note, however, that his simple relationship between myopic and IE learning no longer obtains in next sections.

Statement 2 comes from Desgranges and Gauthier (2003), where it is shown that the MO solution corresponding to the perfect foresight root λ_1 of lowest modulus is the only one to be locally immune to sunspots. This is also in line with Statement 1, given the existing results on susnspot equilibria; see Chiappori, Geoffard and Guesnerie (1992).

Finally, Statement 3 is a consequence of statement 3 in proposition 1, since the "evolutive" learning of growth rates takes place in the one-step forward looking framework of Section 2.

The analysis of IE-stability leading to Statement 4 is provided in Evans and Guesnerie (2003a); see in particular Lemma 1 for a direct proof.[9] It should also be noted that this dynamics is the one obtained by reversing time in the perfect foresight dynamics given in Gauthier (2002).

The fact that for MO converging solutions, the C1 topology on trajectories and the C0 topology on growth rates are equivalent follows for example from Evans and Guesnerie (2003a).

This, together with previous findings, terminates the proof and establishes the present version of then dynamic equivalence principle. ■

This result deserves a number of comments

1. With respect to the model of the previous section, without predetermined variables, statements 1 to 4 are weaker since they all hold locally, and not globally (as all, but Statement 3 of Proposition 1, held before). This comes from the non-linearities that both the perfect foresight dynamics and the learning dynamics on growth rates display.

2. Statement 3 deserves to be emphasized for its strength. The class of "reasonable" learning rules is the same as in Proposition 1, once the state variable of the learning process is reinterpreted as a growth rate, and not a capital level. Naturally, as documented in Evans and Honkapohja (2001), other learning rules, that either are not adaptive or do not detect cycles, may be locally stable. In particular, Evans (1986) shows that the conditions for stability associated with Differential E-Stability, when applied to MO solutions of (7), are weaker than those for IE-stability. This has two consequences: first, many learning rules involving a slow response to new information, the behavior of which is captured by the Differential E-Stability criterion, converge toward the solution we select here, and second, the convergence of these learning rules, viewed as a selection device, selects many more solutions than we do in Proposition 2.

[9] According to this lemma, if at period t all agents conjecture that the growth rate between today and tomorrow is between $\lambda - \varepsilon$ and $\lambda + \varepsilon$ (with $\lambda = \lambda_1$ or $\lambda = \lambda_2$), then the actual growth rate is between $\lambda - (\delta\beta/(1-\beta\lambda)^2\varepsilon + o(\varepsilon^2)$ and $\lambda + (\delta\beta/(1-\beta\lambda)^2\varepsilon + o'(\varepsilon^2)$, where $o(\varepsilon^2)$ and $o'(\varepsilon^2)$ tends to zero with ε^2.

3. The study of eductive stability in Evans and Guesnerie (2003a) shows that IE-stability is a necessary condition of "eductive" stability. But the heterogeneity of agents' behavior and expectations destabilizes learning in a more serious and complex way in the present model (see Proposition 2 of Evans and Guesnerie (2003a)). Also, the fact that agents form their forecasts conditionally on the current stock of capital has strong implications for both "eductive" and IE-stability.

4. This version of the dynamic equivalence principle leads to the selection of a unique stable trajectory in the saddle-point configuration. It also leads the selection of a unique MO solution, even if there exist infinitely many stable trajectories, that is, in the sink configuration. At first sight, this may seem somewhat surprising and worth comments. The reason why λ_1 is the only one to be locally determinate in (17) is that

$$\lim_{t\to+\infty}\frac{k(t)}{k(t-1)}=\lim_{t\to+\infty}\frac{\alpha_1\lambda_1^t+\alpha_2\lambda_2^t}{\alpha_1\lambda_1^{t-1}+\alpha_2\lambda_2^{t-1}}=\lambda_2,$$

   where α_1 and α_2 are two constants determined by the initial condition $k(-1)$ and an arbitrary initial forecast. In other words, as time passes, the growth factor approaches the root of highest modulus along any second order solution, thus making λ_1 locally determinate and λ_2 locally indeterminate in the perfect foresight dynamics of growth rates. Equivalently the λ_1-trajectory is the only one to be determinate in C1 topology of trajectories.

4 Introducing additional lagged variables

4.1 The framework

In this section, we shall extend our analysis to the class of linear univariate models in which there is still only one lead in expectations, but there is now an arbitrary finite number $L \geq 1$ of predetermined lagged variables in each period. The current state of the economic system is then determined by the temporary equilibrium map:

$$\gamma E\left(x\left(t+1\right)\mid I_t\right)+x\left(t\right)+\sum_{l=1}^{L}\delta_l x\left(t-l\right)=0. \tag{23}$$

A perfect foresight equilibrium trajectory is a sequence of levels of the state variable $(x(t), t \geq -L)$ associated with a given initial condition $(x(-1), \ldots, x(-L))$, and such that the forecast $E\left(x\left(t+1\right)\mid I_t\right)$ formed at date t about the next state coincides with the actual state $x(t+1)$ in period $(t+1)$, that is

$$\gamma x\left(t+1\right)+x\left(t\right)+\sum_{l=1}^{L}\delta_l x\left(t-l\right)=0, \tag{24}$$

for $t \geq 0$. In this class of models, the equilibrium law of motion of the level of the state variable can be described by a linear recursive equation that links $x(t)$ to the past history of the system. If, at date t, the current state $x(t)$ depends on the L previous states $(x(t-1), \ldots, x(t-L))$, the equilibrium corresponds to an MO solution. If, on the other hand, it depends on the $(L+1)$ previous states $(x(t-1), \ldots, x(t-L-1))$, then the equilibrium is a bubble solution. Indeed, in this case, the evolution of the economy is determined by the initial condition $(x(-1), \ldots, x(-L))$ and some arbitrary initial forecast $E(x(1) \mid I_0) = x(1)$.

As in the previous sections, we shall continue to focus our attention on the class of MO solutions. The law of motion of the level of the state variable is then described by a linear recursive equation of order L,

$$x(t) = \sum_{l=1}^{L} \bar{\beta}_l x(t-l), \tag{25}$$

in which the vector of coefficients $\bar{\beta} = (\bar{\beta}_1, \ldots, \bar{\beta}_L)$ is to be called a stationary extended growth rate of order L, hereafter denoted stationary EGR(L). As expected the eigenvalues $(\lambda_1, \ldots, \lambda_{L+1})$ of the characteristic polynomial associated with (23) play a key role in the understanding of a stationary EGR(L). Indeed, as shown by Gauthier (2002), the dynamics induced by an MO solution is driven by L of these eigenvalues (and the corresponding eigenvectors of the associated dynamical system). Namely, the coefficient $\bar{\beta}_l$ in (25) is equal to $(-1)^{l+1}\sigma_l(\pounds)$, where $\sigma_l(\pounds)$ represents the lth symmetric polynomial, that is the sum over all the different products of l distinct elements in a set $\pounds$ of L eigenvalues in $(\lambda_1, \ldots, \lambda_{L+1})$.

Subsequently, we shall apply the dynamic equivalence principle to any MO solution, or more exactly to the vector of coefficients $\bar{\beta}$ associated with the solution. The remaining of this section briefly describes how one can apply the above three criteria to a stationary EGR(L).

4.2 Determinacy of extended growth rates

The determinacy device hinges on a new dynamics over vectors of extended growth rates $\beta(t) = (\beta_1(t), \ldots, \beta_L(t))$, derived from the usual dynamics with perfect foresight (24), and whose fixed points are the stationary extended growth rates $\bar{\beta}^m$ $(m = 1, \ldots, L+1)$. In this dynamics, $\bar{\beta}^m$ is locally determinate if and only if there is no induced sequence of extended growth rates such that $\beta_l(t)$ remains arbitrarily close to $\bar{\beta}_l^m$ for any $l = 1, \ldots, L$ and any period $t \geq 0$. Following Gauthier (2002), the extended growth rate perfect foresight dynamics can be obtained by assuming that the law of motion of the level of the state variable satisfies (24), and by imposing in addition the restriction that

$$x(t) = \sum_{l=1}^{L} \beta_l(t) x(t-l), \tag{26}$$

whatever the past history of the state variable $(x(t-l), l=-L,\ldots,t-1)$ and $t\geq 0$ are. Since, by assumption, the law (26) is satisfied in any period $t\geq 0$, we have in particular

$$x(t+1)=\sum_{l=1}^{L}\beta_l(t+1)x(t+1-l). \tag{27}$$

For this value of $x(t+1)$, the state of period t defined by (24) is

$$x(t)=-\sum_{l=1}^{L}\frac{\gamma\beta_{l+1}\left(t+1\right)+\delta_l}{1+\gamma\beta_1\left(t+1\right)}x(t-l), \tag{28}$$

with the convention that $\beta_{L+1}\left(t+1\right)=0$. For (24) and (28) to coincide whatever the past history of the state variable is, it must be the case that

$$\beta_l\left(t\right)=-\frac{\gamma\beta_{l+1}\left(t+1\right)+\delta_l}{1+\gamma\beta_1\left(t+1\right)}, \tag{29}$$

for $l=1,\ldots,L$, and with the convention that $\beta_{L+1}\left(t+1\right)=0$. This is the dynamics we are looking for. It is of the one-step forward looking type, and it has no predetermined variable, that is, it relates $\beta(t)$ to $\beta(t+1)$ at date t. It is shown in Gauthier (2002) that the fixed points of this dynamics are the $(L+1)$ stationary EGR(L).

4.3 Sunspots on extended growth rates

Let us now turn our attention to the construction of sunspot equilibria over extended growth rates. In such equilibria, the vector of extended growth rates is perfectly correlated with sunspots, i.e. $(\beta_1(t),\ldots,\beta_L(t))=(\beta_1(s),\ldots,\beta_L(s))$ if sunspot is s $(s=1,2)$ at date t $(t\geq 0)$. In other words, if agents observe at date t the sunspot signal s, they expect the next extended growth rate to be $(\beta_1(s'),\ldots,\beta_L(s'))$ with probability $\pi(s,s')$, and this belief is self-fulfilling in equilibrium. Actually, if the sunspot signal is s at date t, before observing the actual current state of the economy, they a priori believe that

$$x\left(t\right)=\sum_{l=1}^{L}\beta_l(s)x\left(t-l\right), \tag{30}$$

and they form their forecast about the next level of the state variable conditionally on the current sunspot event:

$$E(x(t+1)\mid I_t)=\sum_{s'=1}^{2}\pi(s,s')\sum_{l=1}^{L}\beta_l(s')x(t+1-l).$$

Let $\bar{\beta}_l(s)=\pi(s,1)\beta_l(1)+\pi(s,2)\beta_l(2)$ for $l=1,\ldots,L$. Then, the actual current state is obtained by introducing this forecast into (29):

$$x\left(t\right)=-\sum_{l=1}^{L}\frac{\gamma\bar{\beta}_{l+1}(s)+\delta_l}{\gamma\bar{\beta}_1(s)+1}x\left(t-l\right), \tag{31}$$

with the convention that $\beta_{L+1}(s) = 0$ whatever s is. The actual law (31) coïhncides with the perceived law (30) for any past history $(x(t-l), l = -L, \ldots, t-1)$ and any $t \geq 0$ if and only if

$$\beta_l(s) = -\frac{\gamma\bar{\beta}_{l+1}(s) + \delta_l}{1 + \gamma\bar{\beta}_1(s)}, \tag{32}$$

for $l = 1, \ldots, L$. A local sunspot equilibrium is an array of S vectors $\beta(s) = (\beta_1(s), \ldots, \beta_L(s))$ of extended growth rates ($s = 1, 2$) such that $\beta(s)$ solves (32), $\beta(s) \neq \beta(s')$ for some s, s', and finally $\beta(s)$ stands arbitrarily close to $\bar{\beta}^m$ for any s (and a given m).

4.4 Learning extended growth rates

Consider the case of a learning scheme in which agents try to discover a stationary EGR(L) through the IE-stability criterion. At time t ($t \geq 0$), their estimate of this vector is $(\beta_1(\tau), \ldots, \beta_L(\tau))$ at outset of step τ ($\tau \geq 0$) of some mental reasoning process. That is, agents a priori believe that the law of motion of the economy is to be given by

$$x(t) = \sum_{l=1}^{L} \beta_l(\tau)x(t-l). \tag{33}$$

Given this conjectured law, agents expect the next state to be

$$E(x(t+1) \mid I_t) = \sum_{l=1}^{L} \beta_l(\tau)x(t+1-l). \tag{34}$$

If agents know how the current state is related to the price forecast and the past history of the system, as summarized by (23), they can deduce that if they form their forecast according to (34), then the current price will be in fact such that

$$x\,(t) = -\sum_{l=1}^{L} \frac{\beta_{l+1}(\tau) - \delta_l}{1 + \gamma\beta_1(\tau)} x(t-l), \tag{35}$$

with the convention that $\beta_{L+1}(\tau) = 0$ whatever τ is. They should accordingly revise their initial guess $(\beta_1(\tau), \ldots, \beta_L(\tau))$ as follows:

$$\beta_l(\tau+1) = -\frac{\beta_{l+1}(\tau) - \delta_l}{1 + \gamma\beta_1(\tau)}, \tag{36}$$

for $l = 1, \ldots, L$. This dynamics is the time mirror of (29), the dynamics with perfect foresight over extended growth rates. The learning dynamics consequently admits the same fixed points as (29): the $(L+1)$ stationary EGR(L). Moreover, a stationary EGR(L) is locally stable if and only if it is locally determinate in (29).[10]

[10] Here, there is no longer an equivalence between IE-stability and myopic learning. In fact, at date t, given a past history $(x(t-1), \ldots, x(t-L))$, agents are not able to infer the actual vector

4.5 The equivalence principle in models with memory

In Gauthier (2002), it is shown that the stationary EGR(L) corresponding to the MO solution driven by $(\lambda_1, \dots, \lambda_L)$ is the only one to be locally determinate in (29). Moreover, Desgranges and Gauthier (2003) show that it is also the only one to be locally immune to sunspots. Finally, since the dynamics with perfect foresight (29) is the time mirror of the learning dynamics (36), it is also the only one to be locally IE-stable.

We have the following result.

Proposition 3. Equivalence principle in one-step forward, memory L, one-dimensional linear systems.

Consider a one-step forward looking model (7) *with L lagged predetermined variable, where $\gamma, \delta_l \neq 0$ for some $l > 0$. Assume that λ_i is real* $(i = 1, \dots, L+1)$. *Then,* MO *solutions exist. Assume also that $|\lambda_1| < \dots < |\lambda_{L+1}|$. The following three statements are equivalent:*

1. A stationary EGR(L) MO *solution is locally determinate in the perfect foresight dynamics of extended growth rates.*

2. A stationary EGR(L) MO *solution is locally immune to (stationary) sunspots.*

3. A stationary EGR(L) MO *solution is locally IE-stable.*

Moreover, there is only one MO *solution satisfying 1 to 3, the one along which the stationary* EGR(L) *is driven by the L perfect foresight roots of smallest modulus* $(\lambda_1, \dots, \lambda_L)$.

Thus, the dynamic equivalence principle is satisfied in a more general setting than the one considered in the previous section, provided that it is suitably applied to the stationary EGR(L), and not to the level, or even the mere growth rate of the level of the state variable. Some additional comments are made.

First, the reader will have noted that this result is weaker than the preceding ones:

1. Previous results about "evolutive" learning do not generalize easily, which may explain why the literature is almost silent on this subject in the context of the present model.

2. We make no statement, although it would be possible to do so, on the relationship between C0 topology on extended growth rates and C1 topology on trajectories.

of extended growth rates $(\beta_1(t), \dots, \beta_L(t))$ from the mere observation of the current state $x(t)$; there is only one (linear) equation, with L unknowns. This highlights that IE-stability is feasible because all the agents are assumed to know the structure of the model, i.e. the precise relation between their forecast and the actual state of the economy.

3. Finally, although we may safely conjecture that IE-stability is a necessary condition of "eductive" stability, we have no results on "eductive" stability in this model.

Second, Statement 1 can be extended in several directions.

1. First, it is shown in Gauthier (2004) that statement 1 holds true in the general configuration where there are $H \geq 1$ leads and $L \geq 1$ lags. Second, it is not required that all the perfect foresight roots be real valued, but only that they have different moduli when they are real. Then, if the MO solution driven by $(\lambda_1, \ldots, \lambda_L)$ does not exist, which is the case in particular if λ_L is complex valued and its conjugate is λ_{L+1}, so that $\sigma_L(\pounds^{L+1}) = \lambda_1 + \cdots + \lambda_L$ is complex valued, there is no locally determinate stationary EGR(L). If, on the other hand, this MO solution does exist, it is the only one to be locally determinate.

2. Concerning the existence of sunspots, some insights about the case where the temporary equilibrium map (23) embodies $H \geq 1$ leads in expectations, but $L = 1$ lag in predetermined variable are given in Gauthier (2003). In this case, the solution along which $x(t) = \lambda_1 x(t-1)$ is again the only one to be locally immune to sunspots. This, together with Statement 2 of the propositiion, suggests that a more general equivalence encompassing Statements 1 and 2 should hold true. However, at this stage, the case in which $H, L \geq 1$ is still open to question.

5 Multidimensional one-step forward looking linear models with memory one

5.1 The framework

We shall consider, (see, for example, Kehoe-Levine (1985) for motivation) a multidimensional linear one-step forward looking economy with one predetermined variable:

$$\mathbf{G}E\left(\mathbf{x}(t+1) \mid I_t\right) + \mathbf{x}(t) + \mathbf{D}\mathbf{x}(t-1) = \mathbf{o}, \tag{37}$$

where $\mathbf{x}$ is a $n \times 1$ dimensional vector, $\mathbf{G}$ and $\mathbf{D}$ are two $n \times n$ matrices, and $\mathbf{o}$ is the $n \times 1$ zero vector. Under the perfect foresight hypothesis:

$$\mathbf{G}\mathbf{x}(t+1) + \mathbf{x}(t) + \mathbf{D}\mathbf{x}(t-1) = \mathbf{o}. \tag{38}$$

A perfect foresight equilibrium is a sequence $(\mathbf{x}(t), t \geq 0)$ associated with the initial condition $\mathbf{x}(-1)$, and such that (38) holds in each period. The dynamics with perfect foresight is governed by the $2n$ eigenvalues λ_i $(i = 1, ..., 2n)$ of the following matrix (the companion matrix associated with the recursive equation (38)):

$$\mathbf{A} = \begin{pmatrix} -\mathbf{G}^{-1} & -\mathbf{G}^{-1}\mathbf{D} \\ \mathbf{I}_n & \mathbf{0} \end{pmatrix},$$

where $\mathbf{0}$ is the n-dimensional zero matrix. In what follows, we shall be interested in the perfect foresight dynamics restricted to a n-dimensional eigensubspace, and especially in the one spanned by the eigenvectors associated with the n roots of lowest modulus. Let by definition $|\lambda_i| < |\lambda_j|$ whenever $i < j$ $(i, j = 1, ..., 2n)$. Let $\mathbf{u}_i$ denote the eigenvector associated with λ_i $(i = 1, ..., 2n)$. Since all the eigenvalues are distinct, the n eigenvectors $(\mathbf{u}_1, ..., \mathbf{u}_n)$ form a basis of the subspace associated with $\lambda_1, ..., \lambda_n$. Let:

$$\mathbf{u}_i = \begin{pmatrix} \tilde{\mathbf{v}}_i \\ \mathbf{v}_i \end{pmatrix}$$

where $\mathbf{v}_i$ and $\tilde{\mathbf{v}}_i$ are of dimension n. It is straightforward to check that if $\mathbf{u}_i$ is an eigenvector, then $\tilde{\mathbf{v}}_i = \lambda_i \mathbf{v}_i$. Let us consider the $2n$-dimensional vector $\mathbf{z}(t+1) = (\mathbf{x}(t+1), \mathbf{x}(t))'$ belonging to the n-dimensional eigensubspace spanned by $\mathbf{u}_1, ..., \mathbf{u}_n$. Its coordinates in the basis are of the following form:

$$\mathbf{z}(t+1) = \begin{pmatrix} \mathbf{a} \\ \mathbf{o} \end{pmatrix}$$

where $\mathbf{a}$ is a n-dimensional vector of coordinates. Therefore, in the canonical basis, we have:

$$\mathbf{P} \begin{pmatrix} \mathbf{a} \\ \mathbf{o} \end{pmatrix} = \mathbf{z}(t+1)$$

where:

$$\mathbf{P} = \begin{pmatrix} \lambda_1 \mathbf{v}_{11} & \cdots & \lambda_2 \mathbf{v}_{21} \\ & & \vdots \\ \mathbf{v}_{1n} & \cdots & \mathbf{v}_{2n} \end{pmatrix} = \begin{pmatrix} \mathbf{V}_1 \mathbf{\Lambda}_1 & \mathbf{V}_2 \mathbf{\Lambda}_2 \\ \mathbf{V}_1 & \mathbf{V}_2 \end{pmatrix}$$

where $\mathbf{V}_1 = (\mathbf{v}_1, ..., \mathbf{v}_n)$, $\mathbf{V}_2 = (\mathbf{v}_{n+1}, ..., \mathbf{v}_{2n})$, and

$$\mathbf{\Lambda}_1 = \begin{pmatrix} \lambda_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \lambda_n \end{pmatrix}, \quad \mathbf{\Lambda}_2 = \begin{pmatrix} \lambda_{n+1} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \lambda_{2n} \end{pmatrix}$$

so that $\mathbf{x}(t+1) = \mathbf{V}_1 \mathbf{\Lambda}_1 (\mathbf{V}_1)^{-1} \mathbf{x}(t)$. Hence, if we pick up some $\mathbf{x}(0)$, then if the n-dimensional subspace is in general position, we can find a single $\mathbf{x}(1)$ in the subspace and generate a sequence $(\mathbf{x}(t), t \geq 0)$ following the just defined dynamics. This generates, according to the terminology of Section 2, a MO solution. Of course, there is no a unique MO solution, since each one of them is associated with a different collection of n different eigenvalues.

The methodology proposed in Sections 2 and 3 can be replicated to obtain MO solutions. Assume that

$$\mathbf{x}(t) = \mathbf{B}\mathbf{x}(t-1) \tag{39}$$

in every period t, and for any n-dimensional vector $\mathbf{x}(t-1)$ ($\mathbf{B}$ is an $n.n$ matrix). Also, $\mathbf{x}(t+1) = \mathbf{B}\mathbf{x}(t)$. Thus, it must be the case that $\mathbf{B} = -(\mathbf{G}\mathbf{B} + \mathbf{I}_n)^{-1}\mathbf{D}$, or equivalently $(\mathbf{GB}+\mathbf{I}_n)\mathbf{B}+\mathbf{D}=\mathbf{0}$. A matrix $\bar{\mathbf{B}}$ satisfying this equation is a stationary extended growth rate. Given law (39), it defines a MO solution. It is shown in Evans and Guesnerie (2003b) that $\bar{\mathbf{B}}=\mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}$, where $\mathbf{\Lambda}$ is a $n\times n$ diagonal matrix whose iith entry is λ_i $(i=1,\dots,n)$ and $\mathbf{V}$ is the associated matrix of eigenvectors. In what follows, we focus attention on the saddle-point case, where $|\lambda_n|<1<|\lambda_{n+1}|$.

5.1.1 The expectational plausibility of MO solutions

Our criteria are the same as before. We will however concentrate on three of them: determinacy, immunity to sunspots, and IE-stability.

Determinacy is viewed through a dynamics of perfect foresight extended growth rates that we analyze first. To this aim, consider

$$\mathbf{x}(t)=\mathbf{B}(t)\mathbf{x}(t-1) \tag{40}$$

where $\mathbf{B}(t)$ is a n-dimensional matrix whose ijth entry is equal to $\beta_{ij}(t)$. If (40) is satisfied whatever t is, it must be the case that

$$\mathbf{x}(t+1)=\mathbf{B}(t+1)\mathbf{x}(t). \tag{41}$$

The dynamics with perfect foresight of the endogenous state variable $\mathbf{x}(t)$ induces a dynamics with perfect foresight of extended growth rates $\mathbf{B}(t)$ that is obtained by introducing (41) into (38):

$$\mathbf{GB}(t+1)\mathbf{x}(t)+\mathbf{x}(t)+\mathbf{Dx}(t-1)=\mathbf{o}$$

$$\Leftrightarrow \mathbf{x}(t)=-(\mathbf{GB}(t+1)+\mathbf{I}_n)^{-1}\mathbf{Dx}(t-1), \tag{42}$$

provided that $\mathbf{GB}(t+1)+\mathbf{I}_n$ is a n-dimensional regular matrix. Given that (40) must be satisfied whatever $\mathbf{x}(t-1)$ is, the EGR perfect foresight dynamics is defined by a sequence of matrices $(\mathbf{B}(t),t\geq 0)$ such that (40) and (42) coïncide, that is:

$$\mathbf{B}(t)=-(\mathbf{GB}(t+1)+\mathbf{I}_n)^{-1}\mathbf{D}\Leftrightarrow(\mathbf{GB}(t+1)+\mathbf{I}_n)\mathbf{B}(t)+\mathbf{D}=\mathbf{0}. \tag{43}$$

This defines the extended growth rates perfect foresight dynamics. Its fixed point are the stationary EGR matrices $\bar{\mathbf{B}}$ such that $\mathbf{B}(t)=\bar{\mathbf{B}}$ in (43) whatever t is.

A sunspot equilibrium on extended growth rates, in the spirit of Section 3, is a situation in which the whole matrix $\mathbf{B}(t)$ that links $\mathbf{x}(t)$ to $\mathbf{x}(t-1)$ is perfectly correlated with sunspots. If sunspot event is s $(s=1,2)$ at date t, then

$$E\left(\mathbf{x}(t+1)\mid s\right)=\left[\pi(s,1)\mathbf{B}(1)+\pi(s,2)\mathbf{B}(2)\right]\mathbf{B}(s)\mathbf{x}(t-1).$$

If so,

$$\mathbf{x}(t)=-\left[\mathbf{G}\left[\pi(s,1)\mathbf{B}(1)+\pi(s,2)\mathbf{B}(2)\right]\mathbf{B}(s)+\mathbf{D}\right]\mathbf{x}(t-1).$$

In a sunspot equilibrium, the a priori belief that $\mathbf{B}(t) = \mathbf{B}(s)$ coïncides with (19) whatever $k(t-1)$ is, that is,

$$\mathbf{B}(s) = -\left[\mathbf{G}\left[\pi(s,1)\mathbf{B}(1) + \pi(s,2)\mathbf{B}(2)\right]\mathbf{B}(s) + \mathbf{D}\right].$$

It remains for us to examine the stability properties of the dynamics with learning according to the IE-stability criterion. At virtual time τ of the learning process, agents believe that, whatever t:

$$\mathbf{x}(t) = \mathbf{B}(\tau)\mathbf{x}(t-1),$$

where $\mathbf{B}(\tau)$ is the τth estimate of the n-dimensional matrix $\mathbf{B}$. Their forecasts are accordingly:

$$E\left(\mathbf{x}_{t+1} \mid I_t\right) = \mathbf{B}(\tau)\mathbf{x}_t.$$

The actual dynamics is obtained by reintroducing forecasts into the temporary equilibrium map (37):

$$\mathbf{GB}(\tau)\mathbf{x}_t + \mathbf{x}_t + \mathbf{D}\mathbf{x}_{t-1} = \mathbf{o} \Leftrightarrow \mathbf{x}_\tau = -(\mathbf{GB}(\tau) + \mathbf{I}_n)^{-1}\mathbf{D}\mathbf{x}_{\tau-1}.$$

As a result, the dynamics with learning is written:

$$\mathbf{B}(\tau+1) = -(\mathbf{GB}(\tau) + \mathbf{I}_n)^{-1}\mathbf{D}. \tag{44}$$

A stationary EGR $\bar{\mathbf{B}}$ is locally IE-stable if and only if the above dynamics is converging when $\mathbf{B}(0)$ is close enough to $\bar{\mathbf{B}}$.

5.1.2 The dynamic equivalence principle

We can state the following proposition:

Proposition 4. Equivalence principle in one-step forward, memory one, multidimensional linear systems.

Consider a stationary EGR *saddle-path like solution (the n smallest eigenvalues have modulus less than 1, the $(n+1)$th has modulus greater than 1).*

The following three statements are equivalent:

1. The EGR *solution is determinate in the perfect foresight growth rates dynamics.*

2. *The* EGR *solution is immune to sunspots, that is, there are no neighbour local sunspot equilibria on extended growth rates with finite support, as defined above.*

3. *The* EGR *solution is locally IE-stable.*

Proof. The equivalence between statements 1 and 3 is proved in Evans and Guesnerie (2003b). It follows, as above, from a time reversion argument.

The equivalence between statements 1 and 2 depends on the development of an argument that relies on Chiappori, Geoffard and Guesnerie (1993). ■

Note that this proposition is still weaker than the previous one.

1. It is concerned with properties of the saddle-path solution, but does not show whether these criteria would select another trajectory in other configurations.

2. In fact, it can be shown that there is only one EGR solution that is determinate, independently of the configuration of the perfect foresight dynamics (see Appendix 3). This solution also satisfies statement 3 and we conjecture that it satisfies in addition statement 2. Obviously, this solution coincides with the saddle-path solution, when the saddle-path solution exists.

3. As previously, Proposition 4 is silent on "evolutive" learning rules, and on the possible connections between topologies on extended growth rates and on trajectories.[11]

   Contrarily to the previous case, here a detailed analysis of "eductive" stability, which highlights the destabilizing effects of heterogeneity, is available in Evans and Guesnerie (2003b) . Still, IE-stability appears as a necessary condition of "eductive" stability.

6 Conclusion

This paper suggests that the so-called "equivalence principle" that stresses the connections among determinacy properties of trajectories, sunspot immunity, "evolutive" and "eductive" learning has a much broader relevance than what a superficial inspection suggests. Still, the generalisation of the principle leads to to shift in its emphasis (the extended growth rates becoming the appropriate reference), a reconsideration of its interpretation (sunspot equilibria relate to extended growth rates) and some degree of weakening of its scope (the connections between "evolutive" learning and other criteria are more difficult to assess). On these grounds, the task undertaken here should be pursued.

Let us finally note that the investigation of the similarities and differences among expectational criteria puts the "eductive" stability viewpoint in an appropriate perspective by showing how it exploits the lines of the standard understanding of "expectational coordination" in dynamical models, and how it allows it to be prolonged and improved upon.

7 Appendix

7.1 Dynamic Equivalence and Bubbles

So far, the dynamic equivalence principle has been applied to the particular class of MO-solutions. It is actually often argued that this class of solutions is the only one to be of economic relevance, in view of the fact that, within this class of solutions,

[11]Evans and Guesnerie's proof (2003b) does however explore some of these relationships

the whole equilibrium trajectory is then determined by the initial conditions. On the contrary, the actual evolution of the system along some bubble solution relies in part on arbitrary forecasts formed in the initial periods, and it is not reasonable to assume that all the agents should succeed in focusing on one particular forecast among infinitely many candidates. In this section, we shall briefly examine whether such bubble solutions would be ruled out the four four criteria under scrutiny in the present paper. We shall use the same methodology as in the previous sections. That is, we shall apply the dynamic equivalence principle to the vector of extended growth rates corresponding to bubble solutions. It should be emphasized, however, that there is no longer a one-to-one correspondence between extended growth rates and perfect foresight solutions. In fact, one EGR now covers a full class of solutions.

For simplicity, the discussion is limited to the simple framework, analyzed in section 3, in which the model embodies one lead in forecasts and one lagged predetermined one-dimensional variable. Recall that the equilibrium law of motion along a bubble solution is described by a second order linear recursive equation

$$k(t) = -\frac{1}{\gamma}k(t-1) - \frac{\delta}{\gamma}k(t-2) \tag{45}$$

for $t \geq 1$. The vector $(-1/\gamma, -\delta/\gamma)$ is a stationary EGR of order 2. In order to derive a perfect foresight dynamics of which it is a fixed point, assume that the capital stock is bound to satisfy, in addition to (45),

$$k(t) = g_1(t)k(t-1) + g_2(t)k(t-2). \tag{46}$$

It is straighforward[12] to verify that (45) and (46) are consistent in each period, and for any past history of economic system, if and only if

$$g_1(t) = -\left[\gamma g_1\left(t+1\right) g_1\left(t\right) + \gamma g_2\left(t+1\right) + \delta\right] \tag{48}$$

and

$$g_2(t) = -\gamma g_1\left(t+1\right) g_2\left(t\right). \tag{49}$$

The two-dimensional dynamics (48)-(49) is not well-defined in the immediate vicinity of $(-1/\gamma, -\delta/\gamma)$. Still, one might argue that $(-1/\gamma, -\delta/\gamma)$ is locally indeterminate in this dynamics since there are infinitely many values of $g_1(t)$ that are consistent with $g_2(t) = -\delta/\gamma$ in (48) and (49); indeed, in this case, $g_1(t+1) = -1/\gamma$ independently of $g_1(t)$.

[12]imposes intertemporal restrictions on the evolution of $(g_1(t), g_2(t))$. Such restrictions obtain as in the previous section. Namely, it follows from (46) that

$$k(t+1) = \left[g_1\left(t+1\right) g_1\left(t\right) + g_2\left(t+1\right)\right] k(t-1) + g_1\left(t+1\right) g_2\left(t\right) k(t-2).$$

Thus, (??) rewrites

$$k(t) = -\left[\gamma g_1\left(t+1\right) g_1\left(t\right) + \gamma g_2\left(t+1\right) + \delta\right] k(t-1) - \gamma g_1\left(t+1\right) g_2\left(t\right) k(t-2), \tag{47}$$

which coincides with (46), for any pair $(k(t-1), k(t-2))$ and any $t \geq 1$, if and only if

In the same way, in a Markovian sunspot equilibrium, $(g_1(t), g_2(t))$ is equal to $(g_1(s), g_2(s))$ if the sunspot event is s at date t. Some components of these vectors must differ according to sunspot events for sunspots to matter. One can verify that the conditions under which agents' beliefs are self-fulfilling can be written, in this case, as:

$$g_1(s) = -\gamma \left[\sum_{s'=1}^{2} \pi(s,s') g_1(s') g_1(s) + \sum_{s'=1}^{2} \pi(s,s') g_2(s') \right] - \delta \tag{50}$$

$$\text{and} \quad g_2(s) = -\gamma \left[\pi(s,1) g_1(1) + \pi(s,2) g_1(2) \right] g_2(s). \tag{51}$$

Suppose now that $g_1(1) = g_1(2) = -1/\gamma$ in (50)-(51), and consider any 2×2 singular Markov matrix $\mathbf{\Pi}$. Then, (50) can be rewritten

$$\pi g_2(1) + (1-\pi) g_2(2) = -\delta/\gamma, \tag{52}$$

where π is any given real number in $[0,1]$, and (51) is always satisfied. It is then easy to show that there exist $(g_1(1), g_1(2), g_2(1), g_2(2))$ arbitrarily close to $(-1/\gamma, -1/\gamma, -\delta/\gamma, -\delta/\gamma)$ and satisfying (50). Indeed, simply notice that (50) holds if $g_2(1) = -\delta/\gamma + \theta/\pi$ and $g_2(2) = -\delta/\gamma - \theta/(1-\pi)$, whatever θ arbitrarily close to 0 is.

To establish the dynamic equivalence principle, it remains to study the properties of myopic and eductive learning. Let us concentrate on the second type of learning scheme. If agents believe that time t ($t \geq 1$) that $(g_1(t), g_2(t))$ should equal $(g_1(\tau), g_2(\tau))$ at some mental step τ ($\tau \geq 1$), then their forecast is

$$E\left[k(t+1) \mid I_t\right] = \left[g_1(\tau)\, g_1(\tau) + g_2(\tau)\right] k(t-1) + g_1(\tau)\, g_2(\tau)\, k(t-2),$$

and the corresponding actual state of the state variable becomes

$$k(t) = -\left[\gamma g_1(\tau)\, g_1(\tau) + \gamma g_2(\tau) + \delta\right] k(t-1) - \gamma g_1(\tau)\, g_2(\tau)\, k(t-2). \tag{53}$$

If agents take into account the feedback effect of their guess $(g_1(\tau), g_2(\tau))$ onto the actual evolution (53), then they should revise their guess as

$$g_1(\tau+1) = -\left[\gamma g_1(\tau)\, g_1(\tau) + \gamma g_2(\tau) + \delta\right], \tag{54}$$

$$g_2(t+1) = -\gamma g_1(\tau)\, g_2(\tau). \tag{55}$$

The time reversion argument applies, since the dynamics (48)-(49) is the time mirror of (54)-(55). Thus, although the eductive learning dynamics is not well defined in the neighborhood of $(-1/\gamma, -\delta/\gamma)$, one may say that this pair of parameters is locally unstable in the dynamics with learning in the sense that there are infinitely many revised beliefs $(g_1(\tau+1), g_2(\tau+1))$ consistent with (54)-(55) when $(g_1(\tau), g_2(\tau))$ does not coïncide with $(-1/\gamma, -\delta/\gamma)$. Hence, there is no reason why agents would revise their guess into $(-1/\gamma, -\delta/\gamma)$ when they make forecast errors.

7.2 Local determinacy of growth rates

The local dynamics of growth rates around a steady growth rate λ_i obtains by linearization of (17) at point $g(t) = g(t+1) = \lambda_i$ $(i = 1, 2)$. One gets, for an arbitrarily small difference $dg(t) = g(t) - \lambda_i$,

$$(\gamma\lambda_i + 1)\, dg(t) + \gamma\lambda_i dg(t+1) = 0 \Leftrightarrow dg(t) = -\frac{\lambda_i}{\lambda_i + 1/\gamma} dg(t+1).$$

Since $\lambda_1 + \lambda_2 = -1/\gamma$, we have finally

$$dg(t) = \frac{\lambda_i}{\lambda_j} dg(t+1)$$

in the immediate vicinity of λ_i $(i = 1, 2)$. A constant growth rate λ_i is locally determinate if and only if $|\lambda_i/\lambda_j| < 1$ (with $i \neq j$, $j = 1, 2$). This is the case if $i = 1$, but not if $i = 2$.

7.3 The Equivalence in the multidimensional model

7.3.1 Local Perfect foresight dynamics of Extended growth rates

Recall that we consider the dynamics (43), namely:

$$\mathbf{GB}(t+1)\mathbf{B}(t) + \mathbf{B}(t) + \mathbf{D} = \mathbf{0}$$

Let $\mathbf{dB}(t) = \mathbf{B}(t) - \bar{\mathbf{B}}$, be arbitrarily small. Then, we have, (see Magnus-Neudecker (1988)):

$$\mathbf{GdB}(t+1)\bar{\mathbf{B}} + (\mathbf{G}\bar{\mathbf{B}} + \mathbf{I}_n)\mathbf{dB}(t) = \mathbf{0}$$

$$\Leftrightarrow \text{vec}\left[\mathbf{GdB}(t+1)\bar{\mathbf{B}}\right] + \text{vec}\left[(\mathbf{G}\bar{\mathbf{B}} + \mathbf{I}_n)\mathbf{dB}(t)\right] = \mathbf{0}$$

$$\Leftrightarrow (\bar{\mathbf{B}}' \otimes \mathbf{G})\text{vec}\mathbf{dB}(t+1) + \left[\mathbf{I}_n \otimes (\mathbf{G}\bar{\mathbf{B}} + \mathbf{I}_n)\right] \text{vec}\mathbf{dB}(t) = \mathbf{0}$$

$$\Leftrightarrow \text{vec}\mathbf{dB}(t+1) = -(\bar{\mathbf{B}}' \otimes \mathbf{G})^{-1} \left[\mathbf{I}_n \otimes (\mathbf{G}\bar{\mathbf{B}} + \mathbf{I}_n)\right] \text{vec}\mathbf{dB}(t)$$

$$\Leftrightarrow \text{vec}\mathbf{dB}(t+1) = -\left[(\bar{\mathbf{B}}')^{-1} \otimes \mathbf{G}^{-1}\right] \left[(\mathbf{I}_n \otimes \mathbf{G}\bar{\mathbf{B}}) + (\mathbf{I}_n \otimes \mathbf{I}_n)\right] \text{vec}\mathbf{dB}(t)$$

$$\Leftrightarrow \text{vec}\mathbf{dB}(t+1) = -\left[((\bar{\mathbf{B}}')^{-1} \otimes \mathbf{G}^{-1})(\mathbf{I}_n \otimes \mathbf{G}\bar{\mathbf{B}}) + ((\bar{\mathbf{B}}')^{-1} \otimes \mathbf{G}^{-1})(\mathbf{I}_n \otimes \mathbf{I}_n)\right] \text{vec}\mathbf{dB}(t)$$

$$\Leftrightarrow \text{vec}\mathbf{dB}(t+1) = -\left[((\bar{\mathbf{B}}')^{-1} \otimes \bar{\mathbf{B}}) + ((\bar{\mathbf{B}}')^{-1} \otimes \mathbf{G}^{-1})\right] \text{vec}\mathbf{dB}(t)$$

$$\Leftrightarrow \text{vec}\mathbf{dB}(t+1) = -\left[(\bar{\mathbf{B}}')^{-1} \otimes (\bar{\mathbf{B}} + \mathbf{G}^{-1})\right] \text{vec}\mathbf{dB}(t)$$

Thus, the EGR perfect foresight dynamics (43) in the neighborhood of a given stationary EGR $\bar{\mathbf{B}}$ is given by $\text{vec}\mathbf{dB}(t+1) = -\left[(\bar{\mathbf{B}}')^{-1} \otimes (\bar{\mathbf{B}} + \mathbf{G}^{-1})\right] \text{vec}\mathbf{dB}(t)$, where $\mathbf{dB}(t)$ is the n-dimensional matrix whose entries are those of $\mathbf{B}(t) - \bar{\mathbf{B}}$ for each component of $\mathbf{B}(t)$ close to the corresponding component of $\bar{\mathbf{B}}$ ($\beta_{ij}(t)$ is close to $\bar{\beta}_{ij}$). The notation vec represents the vectorization of the corresponding matrix and the symbol $\otimes$ denotes the Kronecker (tensorial) product.

7.3.2 Stationary Extended Growth rates

By definition, $\bar{\mathbf{B}} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}$. The eigenvalues of $\bar{\mathbf{B}}$ are the roots of the following characteristic polynomial $\det(\bar{\mathbf{B}} - \theta\mathbf{I}_n) = 0$. Observe now that:

$$\mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1} - \theta\mathbf{I}_n = \mathbf{V}(\mathbf{\Lambda}\mathbf{V}^{-1} - \mathbf{V}^{-1}\theta\mathbf{I}_n)$$

$$\Leftrightarrow \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1} - \theta\mathbf{I}_n = \mathbf{V}(\mathbf{\Lambda}\mathbf{V}^{-1} - \theta\mathbf{V}^{-1}) = \mathbf{V}(\mathbf{\Lambda} - \theta\mathbf{I}_n)\mathbf{V}^{-1}$$

$$\Leftrightarrow \det\left(\mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1} - \theta\mathbf{I}_n\right) = \det\mathbf{V}\det\left(\mathbf{\Lambda} - \theta\mathbf{I}_n\right)\det\mathbf{V}^{-1}$$

$$\Leftrightarrow \det\left(\mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1} - \theta\mathbf{I}_n\right) = \det\mathbf{V}\det\left(\mathbf{\Lambda} - \theta\mathbf{I}_n\right)\frac{1}{\det\mathbf{V}} = \det\left(\mathbf{\Lambda} - \theta\mathbf{I}_n\right)$$

Note now that $(\bar{\mathbf{B}} + \mathbf{G}^{-1}) = \bar{\mathbf{V}}\bar{\mathbf{\Lambda}}\bar{\mathbf{V}}^{-1}$ since $\mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1} + \bar{\mathbf{V}}\bar{\mathbf{\Lambda}}\bar{\mathbf{V}}^{-1} = -\mathbf{G}^{-1}$. This shows that the eigenvalues of $\bar{\mathbf{B}}$ are the ones of $\mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}$, namely n eigenvalues of $\mathbf{A}$ among $2n$. We denote such eigenvalues λ. The eigenvalues of $(\bar{\mathbf{B}} + \mathbf{G}^{-1})$ are the ones of $\bar{\mathbf{V}}\bar{\mathbf{\Lambda}}\bar{\mathbf{V}}^{-1}$, namely the n remaining of $\mathbf{A}$ among $2n$. We denote such eigenvalues $\bar{\lambda}$. Let θ an eigenvalue of $\left[(\bar{\mathbf{B}}')^{-1} \otimes (\bar{\mathbf{B}} + \mathbf{G}^{-1})\right]$. Then we have $\theta = \bar{\lambda}/\lambda$, where $\bar{\lambda}$ is any eigenvalue of $\bar{\mathbf{\Lambda}}$ and λ is any eigenvalue of $\mathbf{\Lambda}$.

The EGR perfect foresight dynamics displays a one-step forward looking structure without predetermined variable. A fixed point of this dynamics, that is a stationary EGR, is accordingly locally determinate if and only if all the n^2 eigenvalues θ (previously defined) have modulus greater than 1. By definition of the eigenvalues λ of $\mathbf{A}$, it must be the case that $\bar{\mathbf{B}}$ is associated with the n-dimensional matrix $\mathbf{\Lambda}$ whose entries are the n eigenvalues of $\mathbf{A}$ of lowest modulus, namely $\lambda_1, ..., \lambda_n$. Hence, as announced, the stationary EGR corresponding to the n eigenvalues $\lambda_1, ..., \lambda_n$ of lowest modulus is the unique stationary EGR that is locally determinate in the extended growth rate perfect foresight dynamics (43).[13]

[13] We do not examine here the connections with the C1 topology of trajectories viewpoint, although the insights of Section should generalize.

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