Macroeconomic and Monetary Policies from the Eductive Viewpoint

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The quality of the coordination of expectations, a key issue for monetary policy, obtains from different, but interrelated, channels: both the credibility of the central bank intervention and the ability of decentralized agents to coordinate on a dynamical equilibrium matter. For both purposes, it is important to understand how agents learn. Indeed, many studies on monetary policy focus on learning processes involving evolutive, real-time learning rules (such as adaptive learning rules).

The eductive viewpoint, as illustrated in Guesnerie (2005) and other references cited herein, partly abstracts from the real-time dimension of learning, with the aim of more directly capturing the systems' coordination-friendly characteristics. The paper first presents the analytical philosophy of expectational coordination underlying the eductive viewpoint. Providing a synthesis of the eductive viewpoint is a prerequisite to comparing the methods that this viewpoint suggests with those actually adopted in most present studies of learning in the context of macroeconomic and monetary policy. Such a comparison rests on the review of existing learning results in the context of dynamic systems, which is currently the main field for applying the eductive method to macroeconomics.[1] Such applications, however, have not had a direct bearing on monetary policy issues. Following the review, the

I thank Carl Walsh for useful comments on an earlier draft and Xavier Ragot for discussions on these issues. I am especially grateful to Antoine d'Autume for pointing out an error in a previous version. Also, section 5 borrows significantly from the joint study of eductive learning in RBC-like models undertaken with George Evans and Bruce McGough (Evans, Guesnerie, and McGough, 2007).

1. See, in particular Evans and Guesnerie (2005); for a static macroeconomic example, see Guesnerie (2001).

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paper explores the differences between the traditional viewpoint and this competing viewpoint as they relate to standard monetary policy analysis. This exploration is tentative, yet promising.

The paper proceeds as follows. The next section lays out the logic behind the eductive viewpoint and compares it with the evolutive approach. I then review results that support a comparison between the most standard expectational criteria and the eductive criterion, first in the framework of a simple one-dimensional dynamic system (section 2) and then in a multidimensional system (section 3). The comparison with standard approaches is completed in section 4. The analysis emphasizes the role of heterogeneity of expectations and may suggest that the alternative view completes and deepens—rather than contradicts—the conclusions of more standard approaches. However, section 5 undertakes an eductive analysis of a simple cashless economy in an infinite-horizon model with infinitely-lived agents, which stresses conditions for expectational coordination that are strikingly different from the classical ones. In particular, the eductive evaluation of the stabilizing performance of the Taylor rule suggests that its reaction coefficient to inflation has to be severely restricted.

1. Expectational Stability: The Eductive Viewpoint

The notion of an eductively stable or strongly rational equilibrium has game-theoretical underpinnings and draws on ideas like rationalizability, dominance solvability, common knowledge. These concepts serve to provide a high-tech justification of the proposed expectational stability criteria. The next subsection emphasizes this high-tech approach for proposing global stability concepts that have a clearly eductive flavor. The local view of the global approach allows a more intuitive, low-tech interpretation which is presented in the second subsection, and the section closes with comments on the connections between the eductive viewpoint and the standard evolutive learning viewpoint.

1.1 Global Eductive Stability

The model assumes rational economic agents (modeled as a continuum), who know the logic of the collective economic interactions (that is, the underlying model). Both the rationality of the agents and the model are common knowledge. The state of the system is denoted E and belongs to some subset ε of some vector space. The state E can be a number, (the value of an equilibrium price or a growth rate), a vector (of equilibrium prices, for example), a function (an equilibrium demand function), an infinite trajectory of states, or a probability distribution. For example, in the variant of the Muth model considered in Guesnerie (1992), E is a number—namely, the market clearing price tomorrow on the wheat market. The agents are farmers whose profits depend on the wheat price. They know the model in the sense that they understand how the market price depends on the total amount of wheat available tomorrow: the market clearing price, as a function of the total crop, is determined from the inverse of a known demand function. Agents know all this, (Bayesian) rationality and the model, and they know that it is known, and they know that it is known that it is known, and so on. With straightforward notation, (it is known)^p for any p (that is, it is common knowledge. In general equilibrium models (Guesnerie, 2001, 2002; Ghosal, 2006), E is a price or quantity vector. In models focusing on the transmission of information through prices (Desgranges, 2000; Desgranges and Heinemann, 2005; Desgranges, Geoffard, and Guesnerie, 2003), E is a function that relates the non-noisy part of excess demand to the asset price. In infinite horizon models, E is an infinite trajectory consisting, at each date t, of either a number or a vector, describing the state of the system at this date. Introducing uncertainty in these partial equilibrium, general equilibrium, and intertemporal models leads to substituting E with a probability distribution over the set of finite or infinite dimensional vectors previously considered.

In this paper, I focus on rational expectations or perfect foresight equilibria. Emphasizing the expectational aspects of the problem, I view an equilibrium of the system as a state, E^*, that prevails if everybody believes that it prevails. Note that in the described context, E^* is such that the assertion, "it is common knowledge that $E = E^*$" is meaningful.

I say that E^* is eductively stable or strongly rational if the following assertion A implies assertion B (given that Bayesian rationality and the model are common knowledge):

Assertion A: It is common knowledge that $E \in \varepsilon$;

Assertion B: It is common knowledge that $E = E^*$.

The mental process that leads from assertion A to assertion B is as follows. First, since everybody knows that $E \in \varepsilon$, everybody knows that everybody limits their responses to actions that are the best responses to some probability distributions over ε. It follows that everybody knows that the state of the system will be in $\varepsilon(1) \subset \varepsilon$. Second, if $\varepsilon(1)$ is a proper subset of ε, the mental process goes on as in the first step, but it is now based on $\varepsilon(1)$ instead of ε. Third, the process continues indefinitely, resulting in a (weakly) decreasing sequence $\varepsilon(n) \subset \varepsilon(n-1) \subset \ldots \subset \varepsilon(1) \subset \varepsilon$. When the sequence converges to E^*, the equilibrium is strongly rational or eductively stable. When convergence does not occur, the limit set is the set of rationalizable equilibria of the model (see Guesnerie and Jara-Moroni, 2007).

Global eductive stability is clearly very demanding, although it can be shown to hold under plausible economic conditions in a variety of models, including partial and general equilibrium standard market contexts (Guesnerie, 1992, 2001), financial models of the transmission of information through prices (Desgranges, Geoffard, and Guesnerie, 2003), and general settings involving strategic complementarities or substitutabilities (Guesnerie and Jara-Moroni, 2007).

1.2 Local Eductive Stability

Local eductive stability may be defined through the same high-tech or hyperrational view. However, the local criterion also has a very intuitive, low-tech, and in a sense boundedly rational interpretation.

1.2.1 Local eductive stability as a common knowledge statement

I say that E^* is locally eductively stable or locally strongly rational if there is some nontrivial neighborhood of E^*, $V(E^*)$, such that assertion A implies assertion B:

Assertion A: It is common knowledge that $E \in V(E^*)$;

Assertion B: it is common knowledge that $E = E^*$.

Hypothetically, the state of the system is assumed to be in some nontrivial neighborhood of E^*, and this hypothetical assumption of common knowledge implies common knowledge of E^*. In other words, the deletion of non-best responses starts under the assumption that the system is close to its equilibrium state. In that sense, this is the same hyperrational view referred to above. However, the statement can be read in a simpler way.

1.2.2 Local eductive stability as a common sense requirement

An intuitively plausible definition of local expectational stability is as follows: there is a nontrivial neighborhood of the equilibrium such that if everybody believes that the state of the system is in this neighborhood, it is necessarily the case that the state is, in fact, in this neighborhood, regardless of the specific form of everybody's belief. Intuitively, the absence of such a neighborhood signals some tendency to instability: there can be facts falsifying any universally shared conjecture on the set of possible states, unless this set reduces to the equilibrium itself. The failure of local expectational stability in the precise sense defined above is (roughly) equivalent to a failure of the local intuitive requirement.

1.3. Eductive versus Evolutive Learning Stability

Milgrom and Roberts (1990) suggest an informal argument according to which, in a system that repeats itself, non-best responses to existing observations will be deleted after a while, initiating a real-time counterpart of the notional-time deletion of non-best responses that underlies eductive reasoning. I focus here on the connections between local eductive stability and the local convergence of standard evolutive learning rules. Local eductive stability, as just defined, implies that once the (possibly stochastic) beliefs of the agents are, for whatever reason, trapped in $V(E^*)$, they will remain in $V(E^*)$ whenever updating satisfies natural requirements that are met in particular by Bayesian updating rules. Although this does not guarantee that any evolutive learning rule will converge, local eductive stability does mean that every reasonable evolutive real-time learning rule will converge asymptotically in many settings (see Guesnerie, 2002; Gauthier and Guesnerie, 2005). Furthermore, the failure to find a set $V(E^*)$ for which the equilibrium is locally strongly rational signals a tendency to trigger away in some cases reasonable states of beliefs that are close to the equilibrium (and are thus likely to be reachable with some reasonable evolutive updating process) a fact that threatens the convergence of the corresponding learning rule.[2]

2. It also forbids a strong form of monotonic convergence.

The very abstract and hyperrational criterion thus provides a shortcut for understanding the difficulties of expectational coordination, without entering into the business of specifying the real-time bounded rationality considerations. Naturally, the eductive criterion is generally more demanding than most fully specified evolutive learning rules (as strongly suggested by the argument sketched above and precisely shown in the previously cited works).

The connection, however, is less clear-cut than just suggested in models with extrinsic uncertainty. In such cases, the equilibrium, as well as a state of the system in the sense of the word used here, is a probability distribution. However, an observation is not an observation on the state in this sense, but information on the state in the standard sense of the word. Evolutive and eductive learning may then differ significantly.

2. Expectational Coordination: Infinite Horizon and One-Dimensional State

Models used for monetary policy generally adopt an infinite horizon approach. This section and the following review existing results on expectational coordination in general and eductive stability in particular, in infinite horizon models. They are based on Gauthier (2003), Evans and Guesnerie (2003, 2005), and Gauthier and Guesnerie (2005). The review will support an expansion of the comparison of the game-theoretical viewpoint stressed in this paper with the standard macroeconomic approach to the problem as reported in Evans and Honkapohja (2001). I start with one-dimensional one-step-forward models with one-period memory.

2.1 The Model

Consider a model in which the one-dimensional state of the system today is determined from its value yesterday and its expected value tomorrow, according to the following linear (for the sake of simplicity) equation:

$$\gamma E\,[x(t+1) \mid I_t] + x(t) + \delta\, x(t-1) = 0,$$

where x is a one-dimensional variable and γ and δ are real parameters ($\gamma, \delta \neq 0$).[3]

A perfect foresight trajectory is a sequence $(x(t), t \geq -1)$ such that

$$\gamma x(t+1) + x(t) + \delta x(t-1) = 0 \tag{1}$$

in any period $t \geq 0$, given the initial condition $x(-1)$.

Assume that the equation $g_1 = -\gamma g_1^2 - \delta$ has only two real solutions, λ_1 and λ_2 (which arise if and only if $1 - \delta\gamma \geq 0$), with different moduli (with $|\lambda_1| < |\lambda_2|$ by definition). Given an initial condition $x(-1)$, there are many perfect foresight solutions, but only two perfect foresight solutions have the simple form

$$x(t) = \lambda_1 x(t-1)$$

and

$$x(t) = \lambda_2 x(t-1).$$

They are called constant growth rate solutions.

The steady-state sequence $(x(t) = 0, t \geq -1)$ is a perfect foresight equilibrium if and only if the initial state $x(-1)$ equals 0. The steady state is a sink if $|\lambda_2| < 1$, a saddle if $|\lambda_1| < 1 < |\lambda_2|$, or a source if $|\lambda_1| > 1$. I focus here on the saddle case, for which the solution, $x(t) = \lambda_1 x(t-1)$, is generally called the saddle path. Economists have long considered this the focal solution, on the basis of arguments that refer to expectational plausibility. The rest of this section reviews the standard expectational criteria that are used and confirms that the saddle-path solution fits them.

2.2 The Standard Expectational Criteria

The standard expectational criteria basically fall into four categories: determinacy, immunity to sunspots, evolutive learning, and iterative expectational stability. I briefly explore each of these in turn and then relate their solutions in an equivalence theorem.

3. Such dynamics obtain, in particular, from linearized versions of overlapping generations models with production, at least for particular technologies (Reichlin, 1986), or infinite horizon models with a cash-in-advance constraint (Woodford, 1994).

2.2.1 Determinacy

The first criterion is determinacy. Determinacy means that the equilibrium under consideration is locally isolated. In an infinite horizon setting, determinacy has to be viewed as a property of trajectories: a trajectory $(x(t), t \geq -1)$ is determinate if there is no other equilibrium trajectory $(x'(t), t \geq -1)$ that is close to it. This calls for a reflection about the notion of proximity of trajectories, that is, on the choice of a topology. While the choice of the suitable topology is open, the most natural candidate is the C_0 topology, according to which two different trajectories, $(x(t), t \geq -1)$ and $(x'(t), t \geq -1)$, are said to be close whenever $|x(t) - x'(t)| < \varepsilon$, for any arbitrarily small $\varepsilon > 0$ and any date $t \geq -1$. In fact, with such a concept of determinacy, the saddle-path solution, along which $x(t) = \lambda_1 x(t-1)$ when $|\lambda_1| < 1 < |\lambda_2|$, is the only solution to be locally isolated—that is, determinate—in the C_0 topology.

In the present context of models with memory, a saddle-path solution is characterized by a constant growth rate of the state variable. This suggests that determinacy should be applied in terms of growth rates, in which case the closeness of two trajectories, $(x(t), t \geq -1)$ and $(x'(t), t \geq -1)$, would require that the ratio $x(t) / x(t-1)$ be close to $x'(t) / x'(t-1)$ in each period $t \geq 0$. This is an ingredient of a kind of C_1 topology, as advocated by Evans and Guesnerie (2003). In this topology, two trajectories, $(x(t), t \geq -1)$ and $(x'(t), t \geq -1)$, are said to be close whenever both the levels $x(t)$ and $x'(t)$ are close, and the ratios $x(t) / x(t-1)$ and $x'(t) / x'(t-1)$ are close in any period.

As emphasized by Gauthier (2002), the examination of proximity in terms of growth rates leads to the analysis of the dynamics with perfect foresight in terms of growth rates. Define $g(t) = x(t) / x(t-1)$ for any $x(t-1)$ and any $t \geq 0$. The perfect foresight dynamics then imply either

$$x(t) = -[\gamma g(t+1)\, g(t) + \delta]\, x(t-1)$$

or

$$g(t) = -[\gamma g(t+1)\, g(t) + \delta]. \tag{2}$$

The perfect foresight dynamics of growth rates then follows from the initial perfect foresight dynamics defined in equation (1). The growth factor $g(t)$ is determined at date t from the correct forecast of the next growth factor $g(t + 1)$. This new dynamics of equation (2) are nonlinear, and they have a one-step-forward-looking structure, without predetermined variables.

The problem has thus been reassessed in terms of one-dimensional one-step-forward-looking models that are more familiar.

2.2.2 Immunity to sunspots on growth rates

Maintaining the focus on growth rates, I now define a concept of sunspot equilibrium, in the neighborhood of a constant growth rate solution. Suppose that agents a priori believe that the growth factor is perfectly correlated with sunspots. Namely, if the sunspot event is $s = 1, 2$, at date t, they a priori believe that $g(t) = g(s)$, that is,

$$x(t) = g(s)\, x(t-1).$$

Thus, their common expected growth forecast is

$$E\,[x(t + 1) \mid I_t\,] = \pi(s,\, 1)\, g(1)\, x(t) + \pi(s,\, 2)\, g(2)\, x(t),$$

where $\pi(s, 1)$ and $\pi(s, 2)$ are the sunspot transition probabilities.

As shown by Desgranges and Gauthier (2003), this consistency condition is written

$$g(s) = -\{\gamma\, [\pi(s,\, 1)\, g(1) + \pi(s,\, 2)\, g(2)]\, g(s) + \delta\}. \tag{3}$$

When $g(1) \neq g(2)$, the formula defines a sunspot equilibrium on the growth rate, as soon as the stochastic dynamics of growth rates are extended:[4]

$$g(t) = -\gamma\, E\,[g(t{+}1) \mid I_t\,]\, g(t) - \delta.$$

2.2.3 Evolutive learning on growth rates

It makes sense to learn growth rates from past observations. Agents then update their forecast of the next period growth rates from the observation of past or present actual rates. Reasonable learning

4. This equivalence relies on special assumptions about linearity and certainty equivalence.

rules in the sense of Guesnerie (2002) and Gauthier and Guesnerie (2005) consist of adaptive learning rules that are able to detect cycles of order two.

2.2.4 Iterative expectational stability

This subsection applies the iterative expectational (IE) stability criterion (see, for example, Evans, 1985; DeCanio, 1979; Lucas, 1978)[5] to conjectures on growth rates. Let agents believe a priori that the law of motion of the economy is given by

$$x(t) = g(\tau)\, x(t-1),$$

where $g(\tau)$ denotes the conjectured growth rate at step τ in some mental reasoning process. They expect the next state variable to be $g(\tau)x(t)$, so that the actual value is $x(t) = -\delta\, x(t-1) / [\gamma g(\tau) + 1]$. Assume that all agents understand that the actual growth factor is $-\delta / [\gamma g(\tau) + 1]$. When their initial guess is $g(\tau)$, they should revise their guess as

$$g(\tau+1) = -\frac{\delta}{\gamma g(\tau)+1}. \tag{4}$$

By definition, IE stability obtains whenever the sequence $(g(\tau), \tau \geq 0)$ converges toward one of its fixed points, a fact that is interpreted as reflecting the success of some mental process of learning (leading to the constant growth rate associated with the considered fixed point). These dynamics are the time mirror of the perfect foresight growth rate dynamics: then, a fixed point λ_1 or λ_2 is locally IE stable if and only if it is locally unstable in the previous growth rate dynamics, that is, in these dynamics, it is locally determinate.

This simple model provides a somewhat careful reminder of the four possible (and more or less standard) viewpoints on expectational stability. I later compare these viewpoints with the so called eductive viewpoint emphasized here. This comparison is facilitated by the fact that these a priori different approaches to the problem select the same solutions, as described in the proposition below.

5. This concept differs from the more usual concept of differential expectational stability (see Evans and Honkapohja, 2001).

2.2.5 An equivalence theorem for standard expectational criteria

Proposition 1. For a one-step-forward, one-dimensional linear model (with one lagged predetermined variable, where $\gamma, \delta \neq 0$), the following four statements are equivalent:

1. A constant growth rate solution is locally determinate in the perfect foresight growth rate dynamics and equivalently here is determinate in the C_1 topology of trajectories.

2. A constant growth rate solution is locally immune to (stationary) sunspots on growth rates.

3. For any a priori given reasonable learning rules bearing on growth rates, a constant growth rate solution is locally asymptotically stable.

4. A constant growth rate solution is locally IE stable.

In particular, a saddle-path solution that clearly meets the first requirement meets all the others. The argument presented in Guesnerie (2002) incorporates earlier findings. For example, the fact that reasonable learning processes converge relies on a definition of reasonableness integrating the suggestions of Grandmont and Laroque (1991) and the results of Guesnerie and Woodford (1991).

Section 4 will compare the standard criteria with the eductive viewpoint on learning, but some game theory flesh will have to be introduced into the model. Before doing that, I focus on a multi-dimensional version of the model.

3. Standard Expectational Criteria in Infinite Horizon Models: The Multidimensional Case

While keeping with one-step-forward-looking linear models with one-period memory, I now turn to the case of a multidimensional state variable.

3.1 The Framework

The dynamics of the multidimensional linear one-step-forward-looking economy with one predetermined variable are now governed by

$$\mathbf{G}E\,[\mathbf{x}(t+1)\,|\,I_t] + \mathbf{x}(t) + \mathbf{D}\mathbf{x}(t-1) = \mathbf{0},$$

where $\mathbf{x}$ is an $n \times 1$ dimensional vector, $\mathbf{G}$ and $\mathbf{D}$ are two $n \times n$ matrices, and $\mathbf{0}$ is the $n \times 1$ zero vector. A perfect foresight equilibrium is a sequence $(\mathbf{x}(t), t \geq 0)$ associated with the initial condition $\mathbf{x}(-1)$, such that

$$\mathbf{G}\,\mathbf{x}(t+1) + \mathbf{x}(t) + \mathbf{D}\mathbf{x}(t-1) = \mathbf{0}. \tag{5}$$

The dynamics with perfect foresight are governed by the $2n$ eigenvalues λ_i ($i = 1,\ldots, 2n$) of the following matrix (the companion matrix associated with the recursive equation):

$$\mathbf{A} = \begin{pmatrix} -\mathbf{G}^{-1} & -\mathbf{G}^{-1}\mathbf{D} \\ \mathbf{I}_n & (\mathbf{0}) \end{pmatrix},$$

where $(\mathbf{0})$ is the n-dimensional zero matrix.

The discussion centers on the perfect foresight dynamics restricted to a n-dimensional eigensubspace, especially the one spanned by the eigenvectors associated with the n roots of lowest modulus. I assume that the eigenvalues are distinct and define $|\lambda_i| < |\lambda_j|$ whenever $i < j$ ($i, j = 1,\ldots, 2n$). I then focus on the generalized saddle-path case, where $|\lambda_n| < 1 < |\lambda_{n+1}|$.

Let $\mathbf{u}_i$ denote the eigenvector associated with λ_i ($i = 1,\ldots, 2n$). Since all the eigenvalues are distinct, the n eigenvectors form a basis of the subspace associated with $\lambda_1,\ldots, \lambda_n$. Let

$$\mathbf{u}_i = \begin{pmatrix} \tilde{\mathbf{v}}_i \\ \mathbf{v}_i \end{pmatrix},$$

where $\mathbf{v}_i$ and $\tilde{\mathbf{v}}_i$ are of dimension n. If $\mathbf{u}_i$ is an eigenvector, then $\tilde{\mathbf{v}}_i = \lambda_i \mathbf{v}_i$.

Hence, on picking up some $\mathbf{x}(0)$, and if the n-dimensional subspace generated by $(\mathbf{u}_1,\ldots, \mathbf{u}_n)$ is in "general position," there is a single $\mathbf{x}(1)$ such that $(\mathbf{x}(0), \mathbf{x}(1)) = \Sigma a_i \mathbf{u}_i$ is in the subspace. This generates a sequence $(\mathbf{x}(t), t \geq 0)$, $(\mathbf{x}(1), \mathbf{x}(2)) = \Sigma a_i \lambda_i \mathbf{u}_i$ following the dynamics defined in equation (5). This generates a solution that is converging in the saddle-path case.

The methodology proposed in the previous section for constructing a constant growth rate solution can be replicated to obtain what is called a minimum-order solution. Assume that

$$\mathbf{x}(t) = \mathbf{B}\,\mathbf{x}(t-1) \tag{6}$$

in every period t and for any n-dimensional vector $\mathbf{x}(t-1)$($\mathbf{B}$ is an $n \times n$ matrix). Also, $\mathbf{x}(t+1) = \mathbf{B}\mathbf{x}(t)$. It must therefore be the case that

$$\mathbf{B} = -\,(\mathbf{G}\mathbf{B} + \mathbf{I}_n)^{-1}\,\mathbf{D},$$

or equivalently

$$(\mathbf{G}\mathbf{B} + \mathbf{I}_n)\,\mathbf{B} + \mathbf{D} = \mathbf{0}. \tag{7}$$

A matrix $\bar{\mathbf{B}}$ satisfying this equation is a minimum-order solution in the sense of McCallum (1983).[6] Gauthier (2002) calls it a stationary extended growth rate. In view of the previous section's analysis of constant growth rate solutions, I use this latter terminology and focus on the expectational stability of extended growth rates.

3.2. The Expectational Plausibility of Extended Growth Rate Solutions According to Standard Criteria

This section concentrates on three of the above criteria: determinacy, immunity to sunspots, and IE stability. Determinacy is viewed through the dynamics of perfect foresight of extended growth rates, which extends the growth rate dynamics previously introduced. For every t, $\mathbf{B}(t)$ is an n-dimensional matrix whose ij^{th} entry is equal to $b_{ij}(t)$ and $\mathbf{x}(t) = \mathbf{B}(t)\mathbf{x}(t-1)$. This matrix is called an extended growth rate (EGR), in line with the terminology of stationary extended growth rates.

Assume that such a relationship holds in all t, so that $\mathbf{x}(t+1) = \mathbf{B}(t+1)\mathbf{x}(t)$; the dynamics with perfect foresight of the endogenous state variable $\mathbf{x}(t)$ imply

$$\mathbf{G}\mathbf{B}(t+1)\,\mathbf{x}(t) + \mathbf{x}(t) + \mathbf{D}\mathbf{x}(t-1) = \mathbf{0},$$

that is,

$$\mathbf{x}(t) = -\,[\mathbf{G}\mathbf{B}(t+1) + \mathbf{I}_n]^{-1}\,\mathbf{D}\mathbf{x}(t-1), \tag{8}$$

provided that $\mathbf{G}\mathbf{B}(t+1) + \mathbf{I}_n$ is a n-dimensional regular matrix.

6. Evans and Guesnerie (2005) show that $\bar{\mathbf{B}} = \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{-1}$, where $\boldsymbol{\Lambda}$ is an $n \times n$ diagonal matrix whose ii^{th} entry is λ_i ($i = 1,\ldots, n$) and $\mathbf{V}$ is the associated matrix of eigenvectors. In what follows, I focus on the saddle-path case, where $|\lambda_n| < 1 < |\lambda_{n+1}|$.

Then, a perfect foresight dynamics of such matrices $\mathbf{B}(t)$ may be associated with a sequence of matrices $(\mathbf{B}(t),\ t \geq 0)$ such that:

$$\mathbf{B}(t) = -[\mathbf{GB}(t+1) + \mathbf{I}_n]^{-1}\,\mathbf{D} \Leftrightarrow [\mathbf{GB}(t+1) + \mathbf{I}_n]\,\mathbf{B}(t) + \mathbf{D} = \mathbf{0}. \qquad (9)$$

This defines the perfect foresight EGR dynamics. Its fixed points are the stationary matrices $\bar{\mathbf{B}}$ such that $\mathbf{B}(t) = \bar{\mathbf{B}}$, in all t. They are solutions of equation (7).

The determinacy of the stationary extended growth rate associated with the matrix $\bar{\mathbf{B}}$, is standardly defined as the fact that $\bar{\mathbf{B}}$ (the infinite trajectory with constant extended growth rate) is locally isolated, that is, that there does not exist a sequence $\mathbf{B}(t)$ of perfect foresight extended growth rates converging to $\bar{\mathbf{B}}$.

A sunspot equilibrium on extended growth rates, in the spirit of the previous section, is a situation in which the whole matrix $\mathbf{B}(t)$ that links $\mathbf{x}(t)$ to $\mathbf{x}(t-1)$ is perfectly correlated with sunspots. If a sunspot event is s ($s = 1, 2$) at date t, then

$$E\,[\mathbf{x}(t+1)\,|\,s] = [\pi(s, 1)\,\mathbf{B}(1) + \pi(s, 2)\,\mathbf{B}(2)]\,\mathbf{B}(s)\,\mathbf{x}(t-1)$$

and

$$\mathbf{x}(t) = -\,\{\mathbf{G}[\pi(s, 1)\,\mathbf{B}(1) + \pi(s, 2)\,\mathbf{B}(2)]\,\mathbf{B}(s) + \mathbf{D}\}\,\mathbf{x}(t-1).$$

In a sunspot equilibrium, the a priori belief that $\mathbf{B}(t) = \mathbf{B}(s)$ is self-fulfilling in all $\mathbf{x}(t-1)$, so that

$$\mathbf{B}(s) = -\,\{\mathbf{G}[\pi(s, 1)\,\mathbf{B}(1) + \pi(s, 2)\,\mathbf{B}(2)]\,\mathbf{B}(s) + \mathbf{D}\}.$$

Finally, the (virtual-time) learning dynamics associated with the IE-stability criterion are as follows. At virtual time τ of the learning process, agents believe that, in all t,

$$\mathbf{x}(t) = \mathbf{B}(\tau)\,\mathbf{x}(t-1),$$

where $\mathbf{B}(\tau)$ is the τ^{th} estimate of the n-dimensional matrix $\mathbf{B}$. Their forecasts are accordingly

$$E\,[\mathbf{x}_{t+1}\,|\,I_t] = \mathbf{B}(\tau)\,\mathbf{x}_t.$$

The actual dynamics are obtained by reintroducing forecasts into the temporary equilibrium map:

$$\mathbf{GB}(\tau)\mathbf{x}_t + \mathbf{x}_t + \mathbf{D}\mathbf{x}_{t-1} = \mathbf{0} \Leftrightarrow \mathbf{x}_\tau = -\ [\mathbf{GB}(\tau) + \mathbf{I}_n]^{-1}\,\mathbf{D}\mathbf{x}_{\tau\text{-}1}.$$

As a result, the dynamics with learning are written

$$\mathbf{B}(\tau + 1) = -\ [\mathbf{GB}(\tau) + \mathbf{I}_n]^{-1}\mathbf{D}. \tag{10}$$

A stationary EGR $\bar{\mathbf{B}}$ is a fixed point of the above dynamics. It is locally IE stable if and only if the dynamics are converging when $\mathbf{B}(0)$ is close enough to $\bar{\mathbf{B}}$.

3.3 The Dynamic Equivalence Principle

The following proposition describes the equivalence principle in one-step-forward, multidimensional linear systems with one-period memory.

Proposition 2. For a stationary EGR, the following three statements are equivalent:

1. The EGR solution is determinate in the perfect foresight extended growth rate dynamics.

2. The EGR solution is immune to sunspots, that is, there are no neighboring local sunspot equilibria on extended growth rates with finite support, as defined above.

3. The EGR solution is locally IE stable.

In particular, the saddle-path solution—which exists when the n smallest eigenvalues of $\mathbf{A}$ have modulus less than 1, with the $(n + 1)^{\text{th}}$ having modulus greater than 1—meets all these conditions.

The proposition, which is proved in Gauthier and Guesnerie (2005), has a flavor similar to that of the one-dimensional case.[7] The connection between evolutive learning and eductive learning is now more intricate, however. It is not as easy to assess the performance of adaptive learning processes in the multidimensional extended growth rates context as in the one-dimensional situation of the previous section: part 3 of proposition 1 has no counterpart here.

7. The equivalence of propositions 1 and 3 follows easily from the above definitions and sketch of analysis. The equivalence with proposition 2 is clearly plausible.

4. Eductive Learning in Dynamic Models

The discussion of eductive learning requires fleshing out the dynamic models under scrutiny with elements from game theory. In other words, the dynamic model needs to be imbedded in a dynamic game. For the sake of completeness, I present the construct proposed in Evans and Guesnerie (2003), which is based on an overlapping generations (OLG) model.

At each period t, there exists a continuum of agents, some of whom react to expectations while others use strategies that are not reactive to expectations (in an OLG context, the latter are in the last period of their lives).[8] The former are denoted ω_t and belong to a convex segment of R, endowed with Lebesgue measure $d\omega_t$. More precisely, agents ω_t have a (possibly indirect) utility function that depends on three factors: their own strategy $s(\omega_t)$; sufficient statistics on the strategies played by others, that is, $\mathbf{y}_t = \mathbf{F}(\Pi_{\omega_t}\{s(\omega_t)\},*)$, where $\mathbf{F}$, in turn, depends first on the strategies of all agents who react to expectations at time t and second on $(*)$, which here represents sufficient statistics on the strategies played by agents who do not react to expectations and includes (but is not necessarily identified with) $\mathbf{y}_{t-1}$; and the sufficient statistics for time $t+1$ as perceived at time t, —that is, $\mathbf{y}_{t+1}(\omega_t)$, which may be random—and also, now directly, the $t-1$ sufficient statistics $\mathbf{y}_{t-1}$.

I assume that the strategies played at time t can be made conditional on the equilibrium value of the t sufficient statistics $\mathbf{y}_t$. Now, let $(\bullet)$ denote both (the product of) $\mathbf{y}_{t-1}$ and the probability distribution of the random variable $\tilde{\mathbf{y}}_{t+1}(\omega_t)$ (the random subjective forecasts held by ω_t of $\mathbf{y}_{t+1}$). Let $\mathbf{G}(\omega_t, \mathbf{y}_t, \bullet)$ be the best response function of agent ω_t. Under these assumptions, the sufficient statistics for the strategies of agents who do not react to expectations is $(*) = (\mathbf{y}_{t-1}, \mathbf{y}_t)$.

The equilibrium equations at time t are written as follows:

$$\mathbf{y}_t = \mathbf{F}\left\langle \Pi_{\omega_t}\{\mathbf{G}[\omega_t, \mathbf{y}_t, \mathbf{y}_{t-1}, \tilde{\mathbf{y}}_{t+1}(\omega_t)]\}, \mathbf{y}_{t-1}, \mathbf{y}_t \right\rangle. \tag{11}$$

8. An agent in period t is different from any other agent in period t', $t' \neq t$. This means either that each agent is physically different or that the agents have strategies that are independent from period to period. In an OLG interpretation of the model, each agent lives for two periods, but only reacts to expectations in the first period of his life.

When all agents have the same point expectations, denoted $\mathbf{y}^e_{t+1}$, the equilibrium equations determine what is called the temporary equilibrium mapping:

$$\mathbf{Q}(\mathbf{y}_{t-1},\mathbf{y}_t,\mathbf{y}^e_{t+1}) = \mathbf{y}_t - \mathbf{F}\{\Pi_{\omega_t}[\mathbf{G}(\omega_t,\mathbf{y}_t,\mathbf{y}_{t-1},\mathbf{y}^e_{t+1})],\mathbf{y}_{t-1},\mathbf{y}_t\}.$$

Also assuming that all $\tilde{\mathbf{y}}_{t+1}$ have a very small common support around some given $\mathbf{y}^e_{t+1}$, decision theory suggests that $\mathbf{G}$, to the first order, depends on the expectation of the random variable $\tilde{\mathbf{y}}_{t+1}(\omega_t)$, which is denoted $\mathbf{y}^e_{t+1}(\omega_t)$ (and is close to $\mathbf{y}^e_{t+1}$). Equation (11), can be linearized around any initially given situation, denoted (0), as follows:

$$\mathbf{y}_t = \mathbf{U}(0)\mathbf{y}_t + \mathbf{V}(0)\mathbf{y}_{t-1} + \int \mathbf{W}(0,\omega_t)\mathbf{y}^e_{t+1}(\omega_t)d\omega_t,$$

where $\mathbf{y}_t$, $\mathbf{y}_{t-1}$, $\mathbf{y}^e_{t+1}(\omega_t)$ now denote small deviations from the initial values of $\mathbf{y}_t$, $\mathbf{y}_{t-1}$, $\mathbf{y}_{t+1}$, and $\mathbf{U}(0)$, $\mathbf{V}(0)$, $\mathbf{W}(0, \omega_t)$ are $n \times n$ square matrices.

If such a linearization is considered only around a steady state of the system, then $\mathbf{y}_t$, $\mathbf{y}_{t-1}$, and so on will denote deviations from the steady state and $\mathbf{U}(0)$, $\mathbf{V}(0)$, $\mathbf{W}(0, \omega_t)$ are simply $\mathbf{U}$, $\mathbf{V}$, $\mathbf{W}(\omega_t)$.

Adding an invertibility assumption yields two reduced forms. First, the standard temporary equilibrium reduced form, associated with homogenous expectations $\mathbf{y}^e_{t+1}(\omega_t) = \mathbf{y}^e_{t+1}$ is

$$\mathbf{y}_t = \mathbf{B}\mathbf{y}^e_{t+1} + \mathbf{D}\mathbf{y}_{t-1}, \tag{12}$$

Second, the stochastic beliefs reduced form is

$$\mathbf{y}_t = \mathbf{D}\mathbf{y}_{t-1} + \mathbf{B}\int \mathbf{Z}(\omega_t)\mathbf{y}^e_{t+1}(\omega_t)d\omega_t, \tag{13}$$

where $\int \mathbf{Z}(\omega_t)d\omega_t = \mathbf{I}$. I use the reduced form in equation (13) to analyze eductive stability.

4.1 Eductive Stability in a One-Dimensional Setting

Based on the above analysis, it seems natural to index beliefs to growth rates. As highlighted in Evans and Guesnerie (2003), beliefs on the proximity of trajectories in the C_0 sense do not have enough grip on the agents' actions. Hence, the hypothetical common knowledge assumption to be taken into account concerns growth rates (the C_1 topology).

(Hypothetical) common knowledge assumption: The growth rate of the system is between $\lambda_1 - \varepsilon$ and $\lambda_1 + \varepsilon$.

Such an assumption about growth rates triggers a mental process that, in successful cases, progressively reinforces the initial restriction and converges toward the solution. The mental process takes into account the variety of beliefs associated with the initial restriction. Common beliefs with point expectations are then a particular case, and it is intuitively easy to guess that convergence of the general mental process under consideration implies convergence of the special process under examination when studying IE stability. This is stressed as such: IE stability is a necessary condition of eductive stability (Evans and Guesnerie, 2003). Proposition 3 then follows from the earlier equivalence theorem (proposition 1):

Proposition 3: If a constant growth rate solution is locally eductively stable or locally strongly rational then it is determinate in growth rates, is locally IE stable, is locally immune to sunspots, and attracts all reasonable evolutive learning rules.

Eductive stability is thus more demanding in general than all the previous equivalent criteria. The fact that it is strictly more demanding is shown by Evans and Guesnerie (2003), although it becomes equally demanding when some behavioral homogeneity condition is introduced.

4.2 Eductive Stability in a Multidimensional Setting

The hypothetical common knowledge assumption to be taken into account naturally has to bear on extended growth rates.

(Hypothetical) common knowledge assumption: The extended growth rate of the system $\mathbf{B}$ belongs to $\mathbf{V}(\bar{\mathbf{B}})$, where $\mathbf{V}(\bar{\mathbf{B}})$ is a neighborhood in the space of matrices (which has to be defined with respect to some distance, normally evaluated from some matrix norm).

As mentioned earlier, if common knowledge of $\mathbf{B} \in \mathbf{V}(\bar{\mathbf{B}}) \Rightarrow \mathbf{B} = \bar{\mathbf{B}}$, then the solution is locally eductively stable or locally strongly rational. As in the one-dimensional case, proposition 4 now follows from proposition 2.

Proposition 4: If a stationary extended growth rate solution is locally eductively stable or locally strongly rational, then it is determinate, locally IE stable, and locally immune to sunspots.

Again, eductive stability is more demanding, in general, than all the standard and equivalent criteria. The reason is that it takes into account the stochastic nature of beliefs and the heterogeneity of beliefs. Both dimensions are neglected explicitly in the iterative expectational stability construct and implicitly in the other equivalent constructs. In fact, as soon as local eductive stability is concerned, point expectations and stochastic expectations may not make much difference (see Guesnerie and Jara-Moroni, 2007). At least locally, the key differences between strong rationality and standard expectational stability criteria stem from the heterogeneity of expectations.

4.3 Standard Expectational Coordination Approaches and the Eductive Viewpoint: A Tentative Conclusion

My comparison of the eductive viewpoint with the standard expectational coordination criteria (determinacy, absence of neighbor sunspot equilibria, and IE-stability) has been limited to the above class of models. An exhaustive attempt would have to extend the class of models under scrutiny in different directions. First, uncertainty (intrinsic uncertainty) would have to be introduced into the models. The analysis should extend, with some technical difficulties, the appropriate objects under scrutiny being respectively probability distributions on growth rates and extended growth rates. The equivalence proposition 2 would most likely have a close counterpart in the new setting. Second, the models would need to incorporate longer memory lags or more forward-looking perceptions (or both). The theory seems applicable, although the concept of extended growth rate becomes more intricate (Gauthier, 2004).

The next set of remarks brings me back to the models used in monetary theory (starting, for example, with Sargent and Wallace, 1975). A number of these models have a structure analogous to the ones examined here, although they often involve intrinsic uncertainty. This suggests two provisional conclusions that will be put under scrutiny in the next section. First, the standard criterion used in monetary theory for assessing expectational coordination, local determinacy, is less demanding than the eductive criterion. This can be seen, within the present perspective, as the reflection of a neglect of a dimension of heterogeneity of expectations that is present in the problem.

Second, the connections between the evolutive and eductive viewpoints are less clear-cut than in the prototype model. The differences have two sources: the theoretical connection between the two types of learning is less well established in the multidimensional case, which often obtains in monetary models of the New-Keynesian type, than in the one-dimensional one (that is, proposition 1-3 has no counterpart in proposition 2); and in a noisy system, agents do not observe, at each step, a state of the system, as defined in the construct (that is, a probability distribution), but a random realization drawn from this probability distribution. Rules on learning, aimed at being efficient, have to react slowly to new information. Intuitively, IE stability and thus eductive stability will be more demanding local criteria than the success of necessarily slow evolutive learning.

However, the above analysis and its provisional conclusions implicitly refer to a true overlapping-generations framework. The equations from which the expectational coordination aspects of monetary policy are most often examined are indeed overlapping, but they come from non-OLG infinite horizon models. Their interpretation within the framework of an eductive analysis should therefore be different.

5. Eductive Stability in a Cashless Economy

The objective here is to introduce very simple versions or models that are used for the discussion of monetary policy and central bank policy. The discussion centers on a simple model of a cashless economy, in the sense of Woodford (2003).

5.1 The Model and the Standard Viewpoint

Consider an economy populated by a continuum of identical agents, who live forever. Each agent α receives $\bar{y}$ units of a perishable good in every period.[9] There is money, and the good has a money price P_t in each period,

The agents have an identical utility function:

$$U = \sum \beta^t u(C_t),$$

where $u(C_t)$ is iso-elastic

9. Although the continuum interpretation continues to hold, the reasoning formally refers to a representative consumer, leaving aside the notation α.

$$u(C_t) = \frac{1}{1-\sigma}(C_t)^{(1-\sigma)}.$$

The first-order conditions are

$$(1+i_t) = \left(\frac{1}{\beta}\right)\left[\frac{u'(C_{t+1})}{u'(C_t)}\right]\left(\frac{P_t}{P_{t+1}}\right)^{-1} = \left(\frac{1}{\beta}\right)\left(\frac{P_{t+1}}{P_t}\right)\left(\frac{C_t}{C_{t+1}}\right)^{\sigma},$$

where i_t is the nominal interest rate.

The central bank decides on a nominal interest rate according to a Wicksellian rule. The rule takes the following form:

$$i_t^m = \phi\left(\frac{P_t}{P_{t-1}}\right),$$

where ϕ is increasing.
The targeted inflation rate is $\Pi^* > \beta$, so that

$$1+\phi(\Pi^*) = \frac{\Pi^*}{\beta}.$$

The money price at time 0 is denoted P_0^*. The targeted price path is

$$P_t^* = P_0^*(\Pi^*)^t.$$

The economy is considered to start at time 1.

The path $P_t = P_t^*$, $C_t(\alpha) = \bar{y}$, $t = 1, 2, \ldots + \infty$, defines a rational expectations (here a perfect foresight) equilibrium, associated with a nominal interest rate $\phi(\Pi^*) = (\Pi^*/\beta) - 1$.

Is this equilibrium determinate? Since all agents are similar and face the same conditions in any equilibrium, any equilibrium has to meet $C_t(\alpha) = \bar{y}$. It follows that any other (perfect foresight) equilibrium $\{P_t'\}$ has to meet

$$\left[1+\phi\left(\frac{P_t'}{P_{t-1}'}\right)\right]\beta = \left(\frac{P_{t+1}'}{P_t'}\right),$$

which can be rewritten, using π_t as the inflation rate:

$$[1 + \phi\,(\pi_t)] = \pi_{t+1}.$$

Any equilibrium close to the stationary equilibrium Π^* would satisfy (with straightforward notation)

$$\phi'(*)\beta\,(\delta\pi_t) = (\delta\pi_{t+1}),$$

an equation incompatible with the proximity of the new equilibrium trajectory to the steady-state trajectory, as soon as $\phi'(*)\beta > 1$. In other words, if $\phi'(*) > (1/\beta)$, then the equilibrium is locally determinate, and this is the condition associated with the Taylor rule (see, for example, Taylor, 1999).

The argument sketched above does not demonstrate that there are no other perfect foresight equilibria outside the neighborhood under consideration, although the one under scrutiny is the only stationary one. Moreover, if the equations are viewed as coming from an OLG framework, I would argue that the equilibrium is locally IE stable, or even here locally eductively stable. Indeed, assume that (a) it is initially common knowledge that inflation will remain forever in the neighborhood of Π^*, and (b) it is common knowledge that a (general) departure of inflation expectations of $\delta\pi_{t+1}$ involves a departure of period t inflation of $\delta\pi_t = (1/\beta\phi')\delta\pi_{t+1}$. The two assertions together imply that the steady-state inflation $*$ is common knowledge. In other words, the equilibrium $*$ is locally eductively stable.[10]

However, assertion (b), which is a core element of the OLG framework, makes no full sense here, where what happens today depends not only on expectations for tomorrow, but necessarily on the whole trajectory of agents' beliefs. To put it in another way, the fact that tomorrow's (period $t + 1$) inflation expectation is π_{t+1} has no final bite on what the equilibrium price may be today in period t. Indeed, an agent's demand in period t as seen from period 1 is

$$C_t(\alpha) = C_1(\alpha)\left\{\beta^{(t-1)/\sigma}\Pi_1^{t-1}\left[(1+i_s)\left(\frac{P_s}{P_{s+1}}\right)\right]^{1/\sigma}\right\}.$$

10. Strictly speaking, the sketched argument only shows that the equilibrium $*$ is locally IE stable. The fact that agents are identical here is more than needed to ensure that heterogeneity of beliefs does not matter, so that IE stability implies eductive stability.

In period *t,* agent α may be viewed as determining its demand as follows. First, take $C_t(\alpha)$ as a starting parameter and compute the infinite sequence,

$$C_{t+\tau}(\alpha) = C_t(\alpha)\left\{\beta^{(t+\tau-1)/\sigma}\Pi_t^{t+\tau-1}\left[(1+i_s)\left(\frac{P_s}{P_{s+1}}\right)\right]^{1/\sigma}\right\}.$$

Then choose $C_t(\alpha)$ so that it meets the consumer's discounted intertemporal budget constraint.

Clearly, such a computation has to be fed by the whole agents' beliefs over the period and not only by their beliefs over the next period! In other words, the connection between t and $t + 1$ emphasized above for the analysis of eductive stability only captures one intermediate step of the choice procedure and not the whole story, as it would in a true OLG framework.

The right question is then the following: if hypothetically it is common knowledge that π_s is close to $\Pi_s^* = \Pi^*$, then is the equilibrium common knowledge? The next section addresses this question.

5.2 Eductive Stability in the Infinite Horizon Cashless Economy: Preliminaries

Consider the world at time 1 and assume that, at the margin of the stationary equilibrium, where the real interest rate is r^*, all agents expect a small departure dr_s, $s = 1,\ldots$. At this stage, it does not matter whether such a departure comes from an expected change in nominal interest rate or an expected change in inflation. Given these changes in beliefs, what is the new first-period equilibrium?

Consumption will not change in period 1. The only adjustment variable is the first period interest rate, which will become $r^* + dr_1$. What will be the equilibrium dr_1? The answer is given by lemma 1.

Lemma 1: The new equilibrium real interest rate is, to the first-order approximation, $r^* + dr_1$, with

$$dr_1 = -\left(\frac{\beta}{1-\beta}\right)(dr_2).$$

Proof: Consider the first-order conditions:

$$C_t(\alpha) = C_1(\alpha)\left[\beta^{(t-1)/\sigma}\Pi_1^{t-1}(1+r_s)^{1/\sigma}\right].$$

Take the log,

$$\log C_t = \log C_1 + \left(\frac{t-1}{\sigma}\right)\log\beta + \left(\frac{1}{\sigma}\right)\sum_1^{t-1}\log(1+r_s),$$

so that, approximately, in the neighborhood of the stationary equilibrium with consumption C^* and interest rate r^* and with $\beta(1 + r^*) = 1$,

$$\left(\frac{dC_t}{C*}\right) = \left(\frac{dC_1}{C*}\right) + \left(\frac{\beta}{\sigma}\right)\left(\sum_{s=1}^{t-1} dr_s\right).$$

Singling out the adjustment variable dr_1,

$$\left(\frac{dC_t}{C*}\right) = \left(\frac{dC_1}{C*}\right) + \left(\frac{\beta}{\sigma}\right)dr_1 + \left(\frac{\beta}{\sigma}\right)\left(\sum_{s=2}^{t-1} dr_s\right).$$

A key remark is that the expected price change only induces a second-order welfare change for the consumer. As is known from consumption theory, the welfare change obtains to the first-order approximation, as the inner product of the price change and of the market exchange vector (the difference between the consumption and the endowment vector).[11] Since this latter vector is zero, the result obtains. Now, the above finding implies that

$$\sum_1^{+\infty}\beta^{t-1}\left(\frac{dC_t}{C*}\right) = 0.$$

I next compute the above expression:

$$\sum_1^{+\infty}\beta^{t-1}\left(\frac{dC_t}{C*}\right) = \left(\frac{1}{1-\beta}\right)\left(\frac{dC_1}{C*}\right) + \left(\frac{1}{\sigma}\right)\left\{\sum_2^{+\infty}\beta^t\left[dr_1 + \left(\sum_{s=2}^{t-1} dr_s\right)\right]\right\}.$$

11. The fact that this is an infinite-commodity setting does not modify the part of the theory under consideration.

In the case of $dr_s = dr_2$, $\forall$s,

$$\sum_{1}^{+\infty}\beta^{t-1}\left(\frac{dC_t}{C*}\right)=\left(\frac{1}{1-\beta}\right)\left(\frac{dC_1}{C*}\right)+\left(\frac{1}{\sigma}\right)\left\{\sum_{2}^{+\infty}\beta^t\left[(dr_1)+\sum_{3}^{+\infty}(t-2)\beta^t(dr_2)\right]\right\}.$$

Because $\sum_{2}^{+\infty}\beta^t = \beta^2/(1-\beta)$, $\sum_{3}^{+\infty}(t-2)\beta^t = \beta^3/(1-\beta)^2$, this implies that

$$\left(\frac{dC_1}{C*}\right)=-\left(\frac{\beta^2}{\sigma}\right)(dr_1)-\left(\frac{\beta^3}{\sigma}\right)(1-\beta)(dr_2).$$

As in equilibrium $dC_1 = 0$, the result follows.

5.3 Eductive Stability: The Core Analysis

As explained above, I implicitly assume that both the model and rationality are common knowledge. Also the monetary rule of the central bank (ϕ) is credibly committed and hence believed. The initial common knowledge restriction has to be a hypothetical restriction on the state of the system. Here the state of the system is entirely defined, once the monetary rule is adopted, by the sequence of inflation rates. Since the equilibrium inflation rate is Π^*, a natural local restriction on beliefs is that the inflation rate is in the range of $[\Pi^* - \epsilon, \Pi^* + \epsilon]$.

Does this belief trigger a collective mental process leading to the general conclusion that $*$ will emerge? The process under discussion takes place in period 1. To illustrate this process, I explore what will happen if in period 1, all agents believe that future inflation will be for ever $\Pi^* + \epsilon$. First, the expected price path will then be $P_t' = P_1(\Pi^* + \epsilon)^{t-1}$, $t = 2, \ldots + \infty$. Second, the expected real interest rate between t and $t + 1$, $t \geq 2$ will be

$$\frac{1 + \varphi(\Pi^* + \epsilon)}{\Pi^* + \epsilon};$$

that is, it will differ from r^* by approximately

$$\left[\frac{1}{(\Pi^*)^2}\right]\left[\varphi'\Pi^* - (1+\varphi)\right]\epsilon.$$

That is,

$$\left(\frac{1}{\Pi^*}\right)\left(\varphi' - \frac{1}{\beta}\right)\epsilon.$$

I assume that in period one, agents make plans contingent on the interest rate (that is, they submit a demand curve). Their conditional inference of the nominal interest rate is then $\varphi(P_1/P_0^*)$.

With regard to their inference of the next period price P_2, $P_2 = P_1(\Pi^* + \epsilon)$.[12] Hence, the expected real interest rate is

$$\frac{1 + \varphi(P_1 / P_0^*)}{\Pi^* + \epsilon};$$

that is, approximately, when writing the first-period inflation rate $(P_1/P_0^*) = (\Pi^* - \epsilon')$,

$$\left(\frac{\varphi'\epsilon'}{\Pi^*}\right) - \left[\frac{(1+\varphi)\epsilon}{(\Pi^*)^2}\right] = \left(\frac{1}{\Pi^*}\right)\left[\varphi'\epsilon' - \left(\frac{1}{\beta}\right)\epsilon\right]$$

Setting $v = \varphi'$ yields the next lemma.

Lemma 2: Under the state of beliefs just considered, the first-period inflation rate is $(\Pi^* - \epsilon')$, where

$$v\epsilon' = \left[\left(\frac{1}{\beta}\right) - \left(\frac{\beta}{1-\beta}\right)\left(v - \frac{1}{\beta}\right)\right]\epsilon$$

Proof: The above formula is applied:

$$dr_1 = -\left(\frac{\beta}{1-\beta}\right)(dr_2),$$

with

$$dr_2 = \left(\frac{1}{\Pi^*}\right)\left(\varphi' - \frac{1}{\beta}\right)\epsilon$$

12. A different assumption on beliefs would be to see the expected price path as $P_t' = P_0^*(\Pi^* + \epsilon)^t$, $t = 2, \ldots +\infty$, so that $P_2^e = P_2^*(\Pi^* + \epsilon)^2$ in period 1.

and

$$dr_1 = \left(\frac{1}{\Pi^*}\right)\left(\varphi'\epsilon' - \frac{1}{\beta}\epsilon\right)$$

If $\varphi' = v$, then

$$v\epsilon' = \left[\left(\frac{1}{\beta}\right) - \left(\frac{\beta}{1-\beta}\right)\left(v - \frac{1}{\beta}\right)\right]\epsilon.$$

This leads to my main result, as presented in the following proposition.

Proposition 5. A necessary condition for the strong rationality of the equilibrium is $(1/\beta) \leq v \leq (1/\beta)[1/(2\beta - 1)]$. Since $1 + r^* = 1/\beta$, the condition can also be written

$$(1+r^*) \leq v \leq \frac{(1+r^*)^2}{(1-r^*)}.$$

Proof: For eductive stability to hold, the initial belief must not be self-defeating. For that, it must be the case that

$$-1 \leq \left(\frac{1}{\beta v}\right) - \left(\frac{\beta}{1-\beta}\right)\left(1 - \frac{1}{\beta v}\right) \leq 1.$$

Take the inequality ≤ 1. It follows that

$$\frac{(1/\beta v)(1-\beta+\beta)}{(1-\beta)} \leq 1 + \left(\frac{\beta}{1-\beta}\right).$$

or

$$\left(\frac{1}{\beta v}\right) \leq 1.$$

Take the inequality $-1 \leq [\ ]$. Then,

$$\frac{(1/\beta v)(1-\beta+\beta)}{(1-\beta)} \geq -1 + \left(\frac{\beta}{1-\beta}\right),$$

or

$$\left(\frac{1}{\beta v}\right) \geq (-1 + 2\beta),$$

or

$$v \leq \left(\frac{1}{\beta}\right)\left(\frac{1}{2\beta - 1}\right).$$

Indeed one conjecture is that this necessary condition is sufficient, as soon as one specifies the initial set of beliefs as avoiding sweeping beliefs (that is, alternating expectations of high and low inflation). In the sense of the general discussion at the beginning of the paper, this is like choosing an appropriate topology for the neighborhood of the steady state (with sweeping beliefs being considered as non-close to the initial one).[13] The proof would consist in showing that the initial beliefs induce a smaller deviation from the targeted inflation, not only in the first period but in any period, and then iterating the argument using the common knowledge assumption.

The result is striking. The range of $v = \varphi'$ that insures eductive stability is rather small. With β close to 1, the condition looks roughly as follows:

$$\left(\frac{1}{\beta}\right) \leq v \leq \left(\frac{1}{\beta}\right)[1 + 2(1 - \beta)].$$

For the sake of illustration, with a high $\beta = 0.95$, this is roughly

$$(1.05) \leq v \leq (1.05)(1.1) = (1.15).$$

More generally, for small r^*, the window for the reaction coefficient is, to the first-order approximation, $[1 + r^*, 1 + 2r^*]$.

The analysis thus suggests that standard Taylor rules are too reactive. Another striking, but not surprising, conclusion is that a plausible intuition within the determinacy viewpoint (that is, the equilibrium is more determinate, and in a sense more expectationally

13. This is reminiscent of the distinction between C_0 and C_1 topology discussed in section 2.

stable, whenever v increases) is plainly wrong here; there is a small window, above $1/\beta$ (and shrinking with β and vanishing when β tends to 1), for expectational stability.

6. Conclusion

Any conclusions are necessarily provisional, since an outsider's random walk in monetary models (albeit starting from a well-established base camp) has to be subjected to criticism. It must also be enriched to develop an intuition that is somewhat missing in the present state of my understanding of the specialized issues that have been addressed. This outsider's walk has, however, attempted to raise interesting questions for insiders and thus open new fronts of thinking.

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