Econometrica, Vol. 70, No. 2 (March, 2002), 439–480

ANCHORING ECONOMIC PREDICTIONS IN COMMON KNOWLEDGE

BY R. GUESNERIE[1]

The paper examines within a unified methodology expectational coordination in a series of economic models. The methodology views the predictions associated with the Rational Expectations Hypothesis as reasonable whenever they can be derived from the more basic Common Knowledge Hypothesis. The paper successively considers a simple non-noisy N-dimensional model, standard models with "intrinsic" uncertainty, and reference intertemporal models with infinite horizon. It reviews existing results and suggests new ones. It translates the formal results into looser but economically intuitive statements, whose robustness, in the present state of knowledge, is tentatively ascertained.

KEYWORDS: Rational expectations equilibria, "eductive" and "evolutive" learning, Common Knowledge.

PART I—PRELIMINARIES: OBJECTIVES, METHODOLOGY, AND EXAMPLE

1. INTRODUCTION

LET ME, IN THE INTRODUCTION, emphasize the three main terms in the title.

First term, *predictions*. Predictions of participants is a key ingredient of social and economic life; broadly speaking, the present analysis attempts to contribute to the discussion of the quality of such predictions. Its more precise concern is well reflected in a sentence of a letter, sent about one century ago, by H. Poincaré to L. Walras: "Vous regardez les hommes comme infiniment égoistes et infiniment clairvoyants. La première hypothése peut êtrè admise dans une premiere approximation, mais la deuxième nécessiterait peut etre quelques réserves."[2] Infinite clairvoyance in Poincaré terms, depicts the quality of "men's" predictions, and according to the most plausible exegesis refers to what is now called the perfect foresight hypothesis, a hypothesis that is implicit in Walras' discussions of intertemporal equilibrium.

If this interpretation is correct, Poincaré's reservations would most likely extend to most models of modern economic theory that assume agents have

[1] I thank several participants of the 1996 regional Econometric Society meetings, and in particular P. A. Chiappori, A. Mas Colell, and J. Moore, for their comments. I am particularly grateful to E. and H. Dierker and B. Grodal for their very careful reading of a first version of the paper and to C. Chamley, J. Davila, and specially G. Desgranges and M. Jerison, for comments on a later version. Thanks are also due to two anonymous readers. The usual disclaimer applies.

[2] The letter is dated Sept. 1901 (see References). My proposed translation is: "You consider men as infinitely selfish and infinitely clairvoyant. The first assumption may be accepted as a first approximation, but the second one may call for some reservations."

rational expectations, a form of "maximal," if not necessarily "infinite," "clairvoyance." Rational expectations is indeed the hypothesis that the present analysis attempts to place under scrutiny. The critical reflections reviewed in this text go along specific lines that we will characterize: they hopefully contribute, together with competing strands of research on the subject, to make explicit and put flesh on "some reservations."

The second term that deserves special emphasis is *economic*. Although many of the basic ingredients of the present analysis have a game-theoretical origin, and relate to the foundations of the concept of Nash Equilibrium, a problem that conceptually encompasses the problem of the foundations of rational expectations, the specific focus of the present paper is economic modelling. Economic models have a logical and mathematical structure that gives them a well defined identity when compared, for example, to the prototype game with two players and finite strategy sets. One of the main objectives of the analysis is to take advantage of the just evoked specificities to discover and stress economic intuition. Indeed, the paper's analysis of expectational coordination refers to Muth, Cournot, Keynes, Walras, and focuses attention on information revelation, OLG models, saddle-path solutions, cycles . . . , rather than on the prisoner's dilemma or the battle of the sexes.[3]

A unified methodology will be precisely defined and justified below and can be somewhat boldly characterized as a Common Knowledge methodology. *Common Knowledge* is indeed the third term I want to emphasize. As a brief and somewhat caricatural justification for the terminology, one may say that originally rational expectations have been viewed as "the extension of the rationality hypothesis to expectations" (Muth (1961, p. 316)) while the present paper and the studies on which it is based attempt to determine when rational expectations can be viewed as "the extension to expectations of the Common Knowledge of rationality hypothesis." In favorable cases, rational predictions are then *anchored* in Common Knowledge.

We can now outline the plan of the paper.

In the remainder of Part 1, the so-called Common Knowledge methodology is described and illustrated in a crude partial equilibrium set-up à la Muth.

The second part of the paper is devoted to the study of expectational coordination, within the Common Knowledge methodology, in a model whose timing of decisions is initially the same as in the Muthian model previously evoked (all agents decide simultaneously), but where the economic aggregate to be predicted is an N-dimensional vector (rather than the one-dimensional "crop" of the Muthian set-up).

The third part first focuses on models in which the outcome is crucially affected by "intrinsic" uncertainty, which bears on the "fundamentals," and proposes

[3] Also, the analysis of the specific contribution of learning studies undertaken within prototype game theoretical frameworks, such as for example, Fudenberg-Levine (1998), to the general purpose of understanding economic expectational stability, is outside the scope of the present paper. See also Marimon (1996).

insights into a variety of models, including models with "information revealing price equilibria."

The end of the third part is an attempt to adapt the present methodology to the study of expectational stability in two categories of simple infinite horizon models.

A more elaborate description of the methodology is now in order.

2. THE METHODOLOGY: ANCHORING PREDICTIONS IN COMMON KNOWLEDGE

We adhere here to a methodology that relies for its foundations on two extreme rationality assumptions: the first one is that economic agents who are the actors of our models are Bayesian-Rational and know the world in which they live ("the model");[4] the second is that both rationality and the "(fundamentals of the) model" are Common Knowledge:[5] every agent is rational and "knows the model" (assertion 1), every agent knows that every other agent is rational and "knows the model" (assertion 2), every agent knows assertion 2 (assertion 3), and so on, ... every agent knows assertion $\tau - 1$, (assertion τ), ... up to infinity. I will sometimes refer to these assumptions, as a joint assumption, the *Common Knowledge Assumption*.

The index τ just introduced, which measures the depth of knowledge, will often be interpreted as measuring virtual time along which the agents' mental processes, aimed at deriving the consequences of each of the infinite sequence of the above assertions, develop.

In the simplest variant of the Muthian model studied in Guesnerie (1992), and which we use here for illustrative purpose, economic agents are farmers; they are rational, in the sense that each of them, anticipating a random selling price for tomorrow's crop (called $\tilde{p}$, the random variable), chooses a crop level that maximizes his expected utility, the expected utility of the sales receipt minus the cost of the crop. Assuming risk neutrality, the crop supplied by farmer i is $S(E(\tilde{p}), i)$, where E designates the expectation operator, and where $S(., i)$ is the standard textbook supply function of farmer i. Price tomorrow is determined deterministically and competitively on a market that equates supply as decided today and exogenous demand associated with the demand function $D(p)$. The Common Knowledge Assumption asserts that the facts just sketched, i.e. individual behavior and its determinants and the market clearing mechanisms, are not

[4] At this stage, the word "model" is somewhat ambiguous and will already become clearer in the introductory example of the next section. Let us say that the word designates the preferences of the agents and the various constraints that the system faces, possibly the market clearing mechanisms. The extent of expectational coordination that may be achieved in the world under consideration is a conclusion, not an assumption of the analysis! Hence the word "model," here, rather means the "fundamentals of the model."

[5] For a thoughtful presentation of some uses of the concept of Common Knowledge in economics, see Geanakoplos (1992).

only known to all the participants but are Common Knowledge (hereafter CK) between them.[6]

Let us go further into the description of the consequences of CK.

The fact that something like the CK Assumption might trigger the *iterative elimination of dominated strategies* à la Luce-Raiffa (1957) has been understood for some time (for an informal argument and pioneering work along these lines, see Moulin (1981)). Recent literature, and in particular Tan-Werlang (1988),[7] has made this point clearer and more rigorous. The CK assumptions just recalled do provide a basic rationale to the different variants of the iterative elimination of strategies that determine *rationalizable solutions* à la Bernheim (1984) and Pearce (1984) or *dominance solvability* à la Farqharson (1969). Our conceptual apparatus, here, closely relates with the just mentioned game-theoretical concepts. In line with the "rationalizability" principles, iterated consequences of the CK assumption are derived and exploited in order to determine predictions of economic agents[8] that, hence, are "anchored in CK."

Our investigation, however, takes advantage of the "economic" context in which the analysis of the whole paper takes place. In most of the models under scrutiny, agents are infinitesimal (as in Muth's model where there is a continuum of farmers), and have a strategy space that is, rather than a finite set, a subset of a vector space (in Muth, the one-dimensional Euclidean space). Also they are concerned not with the detailed profile of the others' strategies but with *an aggregate* that can be viewed, since individual actions have a negligible effect on it, as the aggregate state of the system (the price, or the total crop, in the Muth model). Later, this state will be denoted E, an element of some vector space, on which the equilibrium analysis focuses.[9]

The principles of the proposed approach to coordination can now be presented. They will be repeatedly used throughout the text and hopefully will become progressively clearer.

Besides the initial step that invokes the CK Assumption, three additional steps will be taken:

1. *Pick a "rational expectations" equilibrium and call it* E^*.

Such a rational expectations equilibrium, which in many subsequent cases is indeed a perfect foresight one, may be associated with different mathematical

[6] In fact, as will be clear later, the knowledge (and hence CK) of the individual supply function is not, here, required for the analysis of expectional stability. CK of the aggregate supply function is enough to trigger the results given later.

[7] See also, for a somewhat different intellectual viewpoint, Aumann-Brandenburger (1995).

[8] That are essentially predictions of the actions of others, i.e. that concern, with the terminology of Cass-Shell (1983), "extrinsic" uncertainty.

[9] As already stated, one objective of the paper is to develop economic intuition. Hence, it will avoid as much as possible game-theoretical technicalities and will focus on operational rather than conceptual issues. Remarkable recent references to the ongoing conceptual debate on issues related to rationalizability include: Kajii-Morris (1997) and Morris-Rob-Shin (1995). However, to the best of my understanding, and in view of the specificities of the present framework, their findings would only marginally qualify the analysis of coordination undertaken here.

objects: it may be a Euclidean vector (Part 2), a function (Section 7), a trajectory defined on an infinite number of points in time (Section 8), a random variable describing a "sunspot equilibrium" (Section 8), etc. . . . E^* however belongs to some vector space.

In the simple, non-noisy Muthian model introduced above, the rational expectations equilibrium is associated with a perfect foresight equilibrium price p^*, such that, with straightforward notation:

$$\int S(p^*, i)\, di = \mathscr{S}(p^*) = D(p^*).$$

($\mathscr{S}(.)$ is the standard aggregate supply function of the economy.)

2. *Introduce a "Common Knowledge restriction," or "CK restriction" that places an exogenous bound on the state space and that describes restricted, but CK, beliefs of the agents on the possible states of the system.*

Mathematically, the restriction is associated with a subset $V(E^*)$ of the state space, which in the following will always be a neighborhood of E^*, i.e. that will include an open set containing E^*. In the Muthian model, the restriction applies to the state variable that is supposed to belong to a small interval around p^*. It may be taken symmetric, i.e. with $E^* = p^*$, $V(E^*) = [p^* - \epsilon, p^* + \epsilon]$, where ϵ may be either small or large.

In general, such a CK restriction has two possible interpretations.

First, it may be understood as reflecting the agents' conjectures restricting the strategies played by the others. When economic agents are assumed to be nonatomic, i.e. infinitesimal, their conjectures concerning strategies played by the others turn out to be restrictions on the possible states of the system either today or tomorrow. We assume then that the restrictions on possible outcomes (induced by restrictive conjectures on strategies) are CK. In such a case the CK assumption has to be viewed as *hypothetical*: as explained below, it may appear, under further scrutiny, as, in a sense, inconsistent.

A second possible intepretation of the CK restriction on the possible states of the system is that it is an announced restriction that can be enforced by a Government, through a credible policy. The restriction has then the status of a *credible* restriction. For example, in the Muthian model, credible price restrictions can result from Government commitments that can be implemented through import or export policies (see the discussion in Guesnerie (1992)).

3. *Analyze the consequences of the combination of the CK assumption and the CK restriction on the states of the system.*

The first step of the analysis studies the reactions of the rational agents when their beliefs are in conformity with the initial restriction. Best responses to such beliefs only solicit a subset of strategies or actions and correspondingly generate a subset $\Gamma(V(E^*))$ of the states of the system.

For example, in the Muthian model, a rational agent knowing that the price is between $p^* - \epsilon$ and $p^* + \epsilon$, will supply at least $S(p^* - \epsilon, i)$ and at most $S(p^* + \epsilon, i)$, and will know that total supply will be between $\int S(p^* - \epsilon, i)\, di = \mathcal{S}(p^* - \epsilon)$ and $\int S(p^* + \epsilon, i)\, di = \mathcal{S}(p^* + \epsilon)$. But then the market clearing price will be between $D^{-1}(\mathcal{S}(p^* + \epsilon))$ and $D^{-1}(\mathcal{S}(p^* - \epsilon))$.

Hence $V(E^*) = [p^* - \epsilon, p^* + \epsilon]$ implies $\Gamma(V(E^*)) = [D^{-1}(\mathcal{S}(p^* + \epsilon)), D^{-1}(\mathcal{S}(p^* - \epsilon))]$.

Three cases can occur, in general:

(i) Either some state in $\Gamma(V(E^*))$ does not belong to $V(E^*)$, in which case the initial restriction displays some form of inconsistency,[10] if viewed as hypothetical. If, on the contrary, the restriction is interpreted as a credible Government threat, then, as just emphasized, the threat might not be self-enforcing, even if it were credible.[11]

(ii) $\Gamma(V(E^*))$ is strictly included in $V(E^*)$.

Pursuing the process, using the knowledge of rationality, the knowledge of the knowledge of rationality, and so on, allows determination of $\Gamma^2(V(E^*))$ first, the second iterate of Γ, then $\Gamma^3(V(E^*))$, etc.

(iii) The case, "unlikely," where $\Gamma(V(E^*))$ equals $V(E^*)$, would only be mentioned for the sake of completeness, if it did not belong to a more general set of situations, those where $\Gamma^n(V(E^*))$ converges but not to E^*. In other words, the limit set can then be viewed as the set of "rationalizable" expectations equilibria (this is my terminology; Guesnerie (1992)): it may be "big" (and include the initial rational expectations equilibrium E^* and possibly other rational expectations equilibria). It is also a "curb set" of Basu-Weibull (1991).

Indeed, we are especially interested here in the case *when $\Gamma^n(V(E^*))$ converges to E^*; we then say that the equilibrium is Strongly Rational*[12] *with respect to the restriction $V(E^*)$.*

Naturally, the definition refers to the initial restriction, but one notes that if Strong Rationality obtains from some restriction V, it also obtains for any $V' \subset V$.

Finally, we often restrict attention to the case where $V(E^*)$ *is a "nontrivial" but small*, as small as desired, *neighborhood of the equilibrium state E^**

Under those circumstances, $\Gamma^n(V(E^*))$ has (intuitively) to converge to E^*. Indeed, we say that the equilibrium is *Locally Strongly Rational whenever there*

[10] The careful reader will have noticed that the inconsistency is particularly strong whenever $\Gamma(V(E^*))$ contains $V(E^*)$.

If not, if going beyond first order beliefs, one finds that for some iterate of Γ, let us say Γ^n, $\Gamma^n(V(E^*))$ is contained in $V(E^*)$, then we can still argue, as implicit in the definition below, that the situation is favorable for "eductive learning."

[11] Again, the just made comments restrict attention to "first order beliefs."

Also, modelling the Government as an indirect market participant, would make more explicit the credibility issue. This is left to the reader who should find help in the discussion of Guesnerie (1992).

[12] I use the terminology proposed in Guesnerie (1989, 1992). The equilibrium could also be called "unique rationalizable," or "dominant solvable."

If initially we are in case (i), as strictly speaking allowed by the definition, it is better to refer to some iterate $\Gamma^p(V(E^*))$ as the initial restriction. Alternatively, one might require that all the iterates of Γ be decreasing: $\Gamma^n(V(E^*)) \subset \Gamma^{n-1}(V(E^*))$. Indeed, one might prefer to distinguish these cases more carefully by defining several variants of the concept (see Evans-Guesnerie (1993)). However, for the present text, additional complexity would have little value and would be detrimental to clarity.

exists a small neighborhood $V(E^*)$ of nonempty interior such that $\Gamma^n(V(E^*))$ converges to E^*.

It may also happen that the restrictions imbedded in the initial CK assumption are enough, without further explicit restriction, to trigger convergence to E^*; the equilibrium is called *Globally Strongly Rational*.[13] Naturally, any Strongly Rational Equilibrium (either Globally or not) is, under the assumptions that have been made, Locally Strongly Rational.

In the Muthian model, Local Strong Rationality obtains whenever, for ϵ small enough, $[D^{-1}(\mathscr{S}(p^*+\epsilon)), D^{-1}(\mathscr{S}(p^*-\epsilon))]$ is strictly included in $[p^*-\epsilon, p^*+\epsilon]$ (since then, further iterations will indefinitely restrict the set of CK feasible prices, up to p^*), i.e. whenever the derivative of $D^{-1}\circ S$, evaluated in p^*, is smaller than one, or in other words whenever

$$(\partial S)_* < (-\partial D)_*, \text{ where } (\partial X)_* \text{ denotes the derivative of } X \text{ at } p^*.$$

Under this condition, the slope of the supply function is smaller than the absolute value of the slope of the demand function,[14] so if *the supply elasticity is smaller than the* (*absolute value of the*) *demand elasticity*, *then the equilibrium is Locally Strongly Rational.* See Figure 1.

To come back to the general case and to sum up, the equilibrium is Strongly Rational with respect to the (possibly local) CK restriction $V(E^*)$ (for short, (Locally) Strongly Rational) whenever $\Gamma^n(V(E^*))$ converges to E^*, i.e. whenever the following implication is true:

(α) *It is CK that* $E \in V(E^*) \Longrightarrow$ (β) *It is CK that* $E = E^*$.[15]

We may note here one property associated with Local Strong Rationality of an equilibrium, i.e. the fact that such an equilibrium is necessarily "isolated" or "locally unique." Such a property is trivially satisfied, and in that sense irrelevant to the understanding of the argument, in the just sketched Muth model as well as in most finite horizon models that we consider later. It is however most pertinent for the analysis of Local Strong Rationality (henceforth LSR) in the infinite horizon models of Section 8:[16]

LSR AND DETERMINACY: *LSR of an intertemporal equilibrium* (*with respect to a neighborhood restriction*) *requires that the equilibrium be determinate in the topology in which the neighborhood restriction is stated*.

[13] This is what happens in the linear version of the Muthian model where the knowledge of the model implies the knowledge that the equilibrium price is bounded, a fact that may turn out to be sufficient to trigger convergence of the mental process.

[14] The interested reader will find a detailed discussion of the intuition that underlies the result in Guesnerie (1992). Such an intuition will however be stressed again and even developed within the framework under consideration in the next part of the paper (Part 2), which may be viewed as analyzing an n-dimensional extension of the Muthian framework.

[15] As noted by Tan-Werlang (1988), it is true and straightforward that (β) is always consistent, in the sense considered here, i.e. that $\Gamma(\{E^*\}) = E^*$. This is why the restriction V has to be nontrivial, and why we constrain it to be of nonempty interior with respect to a "reasonable" topology.

[16] Furthermore, the choice of the topology for the definition of the neighborhood may be, as will be seen later, essential.

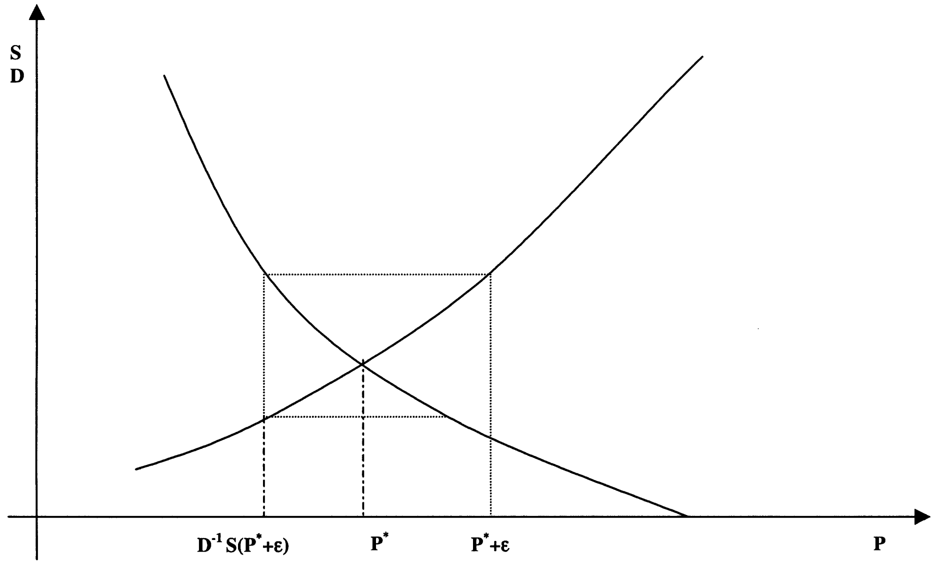


FIGURE 1

PROOF: The proof of the proposition[17] is straightforward once one recalls that a determinate equilibrium has, by definition, no nearby equilibrium. Suppose now that the equilibrium is LSR and indeterminate. That means that in the neighborhood singled out by the neighborhood CK restriction, there is at least one other equilibrium. But this other equilibrium, a Nash equilibrium (in game theoretical terms) that is also compatible with the initial restriction, can never be eliminated in the mental process triggered by the CK assumption. This contradicts the initial hypothesis. *Q.E.D.*

The methodology just sketched allows evaluation of coordination on a given equilibrium, through a test that may be local and based on the CK basic premises. The most convincing justification for the test, and then for the whole methodology, may be to recognize the conceptually damaging effects of the failure of what we called Local Strong Rationality. Let us be more precise. As noted in footnote 15, the assertion: "it is CK that the equilibrium is exactly E^*" is always consistent. But a small "hesitation" or "perturbation" may transform the assertion into: "it is CK that the actual state is very close to (in a small neighborhood of) E^*." Whenever the considered equilibrium is not LSR, the just made "trembling assertion"[18] cannot be self enforcing, whatever the associated "tremble"

[17] It follows from discussions with G. Evans and is also stressed in Evans-Guesnerie (1999).

[18] Game theoretical trembling hand refinements obey a similar, although somewhat more simplistic logic. Most of them can be viewed as associated to different "topologies," in our terminology, on finite dimensional trembles.

neighborhood—at least if it is of nonempty interior. In a sense, the assertion is inconsistent. On theoretical grounds, such an inconsistency seems to be a particularly undesirable property of an equilibrium prediction claiming to be grounded in "rationality."

The test may then be viewed as a robustness test, which serves as a (not necessarily local) refinement device among the rational expectations equilibria. Both the initial restriction on which it is based and the mental process that is triggered have an interesting economic content that has to be ascertained in each case.

PART II—An N-Dimensional Model with a Muthian Decision Structure

This part is devoted to the study of a model that has a timing decision structure analogous to the one of the Muthian model—where farmers' decisions "today," are based on the anticipated market clearing price and hence on the expectations of the crop "tomorrow" (which itself results from all simultaneous decisions "today")—but involves an N-dimensional, rather than one-dimensional, state space.

We are going first to introduce the model, its equilibria, and standard crude analysis of their "expectational" stability (Section 3). Then, following Evans-Guesnerie (1993), we shall develop the rigorous analysis of Strong Rationality and emphasize the economic intuition of the results (Section 4). We shall finally relax the assumption of simultaneity of decisions, which is imbedded in the initial reduced form model, and show how sequentiality increases the chances of sophisticated expectational stability (Section 5).

3. The Reduced Form Model: Existence of Equilibria, "Standard" Stability Concepts

3.1. *The Reduced Form Model*, *Existence and Uniqueness*

As announced, we are going to introduce an N-dimensional model that allows attainment of a significant generalization of the coordination argument, a simple version of which was exhibited in the Muthian model.

As in the Muthian model, we consider a continuum of agents who have to decide now, *simultaneously*, and who are concerned with the *consequences* of the actions decided today on the state of the system *tomorrow*. Indeed, each agent has two different concerns: he is concerned about his *own action* and also about *an N-dimensional aggregate.* Indeed, this N-dimensional aggregate, a vector denoted X, results (possibly in a complex way) from the actions taken by the others.[19]

Even if one does not describe in detail how the aggregate vector is generated from individual interactions,[20] we are still in a position to concentrate on a

[19] Including himself, but he is infinitesimal so that his own action is insignificant for the aggregate action.

[20] A possible connection is described carefully in Evans-Guesnerie (1993) to whom the interested reader is invited to refer.

reduced form of the model. Within such a reduced form, it is enough to describe how *point expectations* bearing on the aggregate, which describe *deterministic* beliefs *identical across all the agents*, determine its actual value, i.e. the actual (aggregate) state of the system.

Let then Ψ be the mapping that describes the effect of changes of point expectations on the system and associates the (N-dimensional) aggregate state of the system $\Psi(X)$ to the (N-dimensional) vector of point expectations X (common to all agents).

In particular, we know from Ψ that if expectations switch from X to X^e, with X^e close to X, then, to a first order approximation, the system switches to X', with

$$X' - \Psi(X) = A(X)(X^e - X), \tag{3.1}$$

where $A(X)$ is the $N \times N$ Jacobian matrix $\partial\Psi/\partial X$, of the mapping Ψ, evaluated in X.[21]

All relevant information for determining Self Fulfilling (i.e. Perfect Foresight or Rational Expectations) predictions of the aggregate state of the system, is provided by the mapping Ψ: A *Rational Expectations Equilibrium* (for the aggregate state of the system) $\overline{X}$ is a fixed point of Ψ, i.e. satisfies

$$\overline{X} = \Psi(\overline{X}). \tag{3.2}$$

Existence and Uniqueness Theorems for the problem under consideration are standard. I shall present them voluntarily in the loose form of an informal statement.

EXISTENCE AND UNIQUENESS OF RATIONAL EXPECTATIONS EQUILIBRIA:
(i) *Under "appropriate" boundary behavior, an Equilibrium* $\overline{X}$ *exists.*
(ii) *If for each* X *in a "broad enough" domain, the matrix* $I - A(X)$ *has a positive determinant, then the equilibrium is unique.*

A fully precise statement would obtain, together with a proof, by assuming that Ψ is continuously differentiable. The "appropriate" boundary behavior is the one that guarantees that the vector field $X - \Psi(X)$ points "outwards," for example on the boundary of some set (hypercube or hypersphere) of R^N. The "broad enough" domain is one where all equilibria could a priori be located (that may be the whole subset whose boundary was just under consideration). With the assumption on the determinant of the matrix $I - A$, uniqueness is a consequence of the Poincaré-Hopf theorem.[22]

[21] We know from Hartman's theorem that the validity of the linear approximation for the local study of the nonlinear dynamics is limited to the case where $A(X)$ has no eigenvalue of modulus one. All the local analysis of the next sections is made under this assumption.

[22] If the vector field "points outwards," then the Poincaré-Hopf theorem says that the sum of the indices, taken in each zero of the vector field, is (+1).

The index is +1 (resp. −1) if $\det(I - A)$ is positive (resp. negative). Hence the conclusion.

We leave to the reader discussion in more detail of the problem of the structure of the matrix A[23] and a sketch of more explicit sufficient uniqueness conditions.[24]

3.2. *A Crude Local Analysis of Expectational Coordination*: *The Iterative Expectational Viewpoint*

The analysis of this subsection takes place in a neighborhood of a Rational Expectations Equilibrium; let us call it $\overline{X}$. For the sake of notational simplicity, we denote $A(\overline{X})$ as just A, and we assume any required differentiability together with the local regularity condition of footnote 21.

Let us consider the viewpoint of an outside observer who assumes the following:

(i) All agents have identical point expectations that, furthermore, lie in some small neighborhood of $\overline{X}$; call it V.

Under such an assumption, one can predict that, to the first order approximation, the aggregate state of the system will lie in a set $\Psi(V)$, close to $\overline{X}+AV$, where AV is the image of the neighborhood through the linear mapping A.

Assume now, for example because the observer's prediction has become known from the agents in the system, the following:

(ii) All agents have identical point expectations that lie in the set $\Psi(V)$.

Then, one can predict that (the aggregate state of) the system will lie in $\Psi(\Psi(V))=\Psi^2(V)$, where Ψ^2 designates the second iterate of Ψ. This latter set is close to the set $\overline{X}+A^2V$.

...

Finally, iterating the argument n times, one concludes that the system lies in the set $\Psi^n(V)$, whatever n and this set is close to $\overline{X}+A^nV$.

Clearly, the justification of the iterative process that has been just presented, or any variant of such a justification that the reader might prefer, is conceptually ambiguous. It is based, at least in the interpretation that is stressed here, on a

[23] For example, if the number a_{ii} is positive (resp. negative), an increase of X_i^e, the (identical through agents) point expectation of the ith component of X_i, generates a positive (resp. negative) reaction of this ith component. We have *strategic complementarities* (resp. *strategic substitutabilities*) *along the* ith *dimension*: an increase of optimism, along this dimension, amounts to a positive (resp. negative) shock on the system.

When all the numbers a_{ij} are positive (resp. negative), we may say that we have *generalized strategic complementarities* (resp. *strategic substitutabilities*) in the sense that increased optimism on any component of the state variable induces a positive (resp. negative) shock on any other component. As we shall emphasize later, when all a_{ij} are positive, our model is (very close to being) a special case of the models considered in the strategic complementarity literature (for an early influential contribution, see Topkis (1979)).

Finally, we may also associate the idea of complementarity and substitutability, as we do later, with the sign of the real part of the eigenvalues of A.

[24] In the case that was just termed of generalized strategic complementarities, the matrix A is a positive matrix. Then $I-A$ has a positive determinant whenever it is an M-matrix (eigenvalues have a positive real part).

In the case where A has only negative components (generalized strategic substitutabilities), the same property holds true under a broad set of conditions.

kind of thought process whose exact status is debatable and whose foundations are unclear: why should it be the case that agents have identical expectations; why should they be point expectations?[25]

Whatever may be, following Evans (1985), let us say the following.

DEFINITION 1: The equilibrium is (locally) *Iteratively E-Stable* (from now on *IE-Stable*) whenever for V close enough to $\overline{X}$, the set $\Psi^n(V)$ converges uniformly to $\overline{X}$.[26]

From the argument sketched above, it appears that the set $\Psi^n(V)$ is close to the set $\overline{X} + A^n V$. Hence we have the following:

IE-STABILITY: *Local IE-Stability is essentially equivalent to the following fact: All eigenvalues of A have modulus strictly smaller than one.*

In all that follows, one will assume that the matrix is semi-simple, a "generic" property in an appropriately defined space of matrices. A semi-simple matrix has N distinct (real or complex) eigenvectors that allow generation of a basis of R^N (that we shall call from now the "eigenvectors basis").[27] Hence, and this is the property of interest to us, the modulus of the eigenvalue(s) of maximal modulus[28] equals *the norm of the matrix induced by the Euclidean vector norm in the just defined basis.*

Hence the property can be read: *the norm of the matrix, "induced" by the Euclidean norm in its own basis, is smaller than one.*

As presented here, IE Stability is related to an argument that may be called "eductive." There are close connections, in this model as in some others discussed below, between "eductive" and "evolutive" stories. Some of them are stressed in the next subsection devoted to "learning stability."

3.3. *Expectational Coordination and Learning Stability*

The above presentation of IE Stability has been based on "eductive" considerations. An equivalent "evolutive" story might have been presented:

Assume that the system was initially (at time 1) in x_1, and that the agents believe, as in a Cobweb tatonnement, that the present values of the variables (at

[25] Indeed, the criticism sketched here has been made to the early propositions of such criteria by Lucas, De Canio, Evans (see, for example, Calvo (1983)).

[26] When V is not a neighborhood of $\overline{X}$, the criterion is global instead of local: we have *Global IE Stability*.

[27] See Evans-Guesnerie (1993) for detailed definitions. Also, considering this class of matrices intends to simplify statements, since the properties stressed here "essentially" extend by continuity.

[28] Indeed, to any vector norm [.] on R^N, there is an associated (or "induced") matrix norm defined as:

$$\|A\| = \mathrm{Sup}_{X/[X]=1}[AX].$$

Not all matrix norms are of this form (see Horn-Johnson (1985)).

1) are accurate predictors of the future (at time 2). Then, at time 2, the system is in $x_2 = \Psi(x_1)$. If the simple forecasting rule (in spite of its repeated failures) is repeatedly used, then at time $n+1$, the system will be in $\Psi^n(x_1)$, where Ψ^n designates the nth iterate of Ψ... etc....

(Local) IE stability can then be viewed as a (local) stability criterion when a "Cobweb-like" evolutive learning rule is used by all the agents of the system.

A further investigation of the relationship of the restrictions involved by IE-Stability with the convergence conditions of truly "evolutive" learning rules, leads us to consider the class of Componentwise Adaptive Learning Rules (CALR):

DEFINITION 2 (*CALR*): *CALR* are as follows:

$$x^e_{\tau,\tau+1;k} = \alpha_k x_{\tau,k} + (1-\alpha_k)x^e_{\tau-1,\tau;k}, \quad \text{for all } \tau \text{ and } k \quad (0 \le \alpha_k \le 1),$$

where x designates the state variable either expected (with superscript e) or realized (without the superscript), where the last subscript k designates the kth coordinate of x, and the first subscript t designates time; where the subscripts τ, τ' mean that the expectation is formed at time τ and concerns time τ'.

Clearly CALR is a standard adaptive rule that says that, in each coordinate, the expectations' correction is a fraction, that may depend on the coordinate (it is α_k) of the mistake that was previously made. Asymptotic Learning Stability obtains whenever the process CALR converges to the equilibrium. We are interested in Local Stability, i.e. in stability subject to the fact that initial expectations are close to equilibrium expectations. Let us take the viewpoint of an outside observer who knows that CALR is used but who does not know the corrections coefficients α_k that are used. When is it the case that the learning rules CALR are locally asymptotically stable, whatever the correction coefficients (between zero and one) chosen?

A (partial) answer is as follows:

ASYMPTOTIC STABILITY OF CALR: *A necessary condition for the local asymptotic stability of all CALR learning rules is that the norm of A, induced by the Euclidean norm in its own basis, is smaller than* 1. *A sufficient condition, when all the corrections coefficients are equal, is that the norm of A, induced by the Euclidean norm in the canonical basis of R^n, is smaller than one.*

The fact that the condition is necessary has already been proved: the "Cobweb-like" rule belongs to the class and its local stability is equivalent to IE Stability.

Sufficiency is shown in the footnote.[29] As the norm of A induced by the Euclidean norm in the canonical basis of R^n is larger than the norm of A similarly

[29] We consider the linearized system. Coming back to the vector notation and reminding that $x_t = Ax^e_{t-1,t}$, expectations governed by the CALR rule evolve as

$$x^e_{t,t+1} = (\bar{\alpha}A + I - \bar{\alpha})x^e_{t-1,t}$$

where $\bar{\alpha}$ designates the diagonal matrix with diagonal elements α_k.

If $\|.\|$ is any norm: $\|\bar{\alpha}A + I - \bar{\alpha}\| \le \|\bar{\alpha}\|\|A\| + \|I - \bar{\alpha}\|$. If $\|.\|$ is the norm (induced by the Euclidean induced in its own basis, the sufficiency condition is strictly more demanding than the necessary condition even when the "correction" coefficients α_k are identical. Only when $n = 1$, the two conditions coincide (as noted in Guesnerie (1992)). The relationships between, on the one hand, the conditions for IE-Stability and, on the other hand, the conditions of asymptotic stability of the class of learning rules under consideration here, are hence much looser in the N-dimensional case than they are in the one-dimensional case. They are still not foreign to each other and it will be interesting to compare them to the conditions for Strong Rationality.

4. (LOCAL) STRONG RATIONALITY OF EQUILIBRIUM: THE ROLE OF HETEROGENEITY OF EXPECTATIONS

4.1. *Local Strong Rationality*

The viewpoint taken in this paper, and presented and advocated above, leads us to analyze expectational coordination from more basic assumptions. Our starting point for the analysis of Local Strong Rationality of the equilibrium is the hypothesis that "it is Common Knowledge that the aggregate state of the system, here the vector, X, lies in a neighborhood of its equilibrium value $\overline{X}$." Indeed this neighborhood is of nonempty interior in R^N.

This assumption implies that each individual agent's expectations have to lie in the set under consideration; however, these expectations have no reason, in general, to be point expectations and/or to be identical across agents. Hence, we cannot go further without knowing how the system reacts to stochastic expectations that are heterogeneous across agents. The description of such reactions is here the subject of an axiom, whose field of validity covers, to the best of my understanding, most economic models:

AXIOM 1: *Assume that each individual agent i's expectations of the state of the system are described by a random variable $\tilde{\tilde{X}}(i)$, whose support is in V, a neighborhood of X, and whose mean is $X(i)$.*

Then, if V is small enough, the state X' of the system is, to the first order approximation,

$$X' - \Psi(X) = \int A(i, X)[X(i) - X]\,di. \tag{4.1}$$

Here,[30] the $A(i, .)$ are $N \times N$ matrices (that depend upon X, although for notational simplicity, we shall drop it later) that generally differ across agents. The

norm) in the canonical basis of R^n, the right-hand side equals

$$(\max \alpha_k)\|A\| + (1 - \min \alpha_k).$$

Hence, when $\max \alpha_k = \min \alpha_k$, the conclusion follows.

[30] Also, we shall always adopt the normalization that the size of the population is $\int di = 1$. As we shall see later, the axiom has some connections with the consistency of derivatives axiom of Guesnerie (1986) and Chiappori-Guesnerie (1989).

axiom says that the effects of individual expectations on the state of the system are additive across agents (a typical first order feature); the contribution of each agent is associated with a personalized matrix $A(i)$ and only depends on *the discrepancy between the mean of the agent's expectations and the reference value.* The fact that only the mean intervenes is crucial and reflects a first order approximation effect that holds true, in particular, in optimizing models where decisions under uncertainty are made by expected utility maximizers.

We note here that Axiom 1 describes (local) changes of the system associated with changes in expectations in a setting that is more general than the setting of Section 3.1, where similar changes were evaluated under the assumption that all agents had identical point expectations. The consistency of Axiom 1 with the previous formula has then to be checked.

First, we note that if all agents have identical point expectations X, Axiom 1 tells us that $X' = \Psi(X)$, as in the previous discussion.

Second, we note that when agents have identical point expectations, $X^e \neq X$ (as was previously assumed), then the right-hand side of (4.1) reduces to

$$\left[\int A(i)\, di\right][X^e - X].$$

Comparing with previous formula (3.1), this means that $\int A(i)\, di$ *is necessarily equal to the previous matrix* A: $\int A(i)\, di = A$.

With the just presented conceptual apparatus, we are in a position to assess a sufficient condition for Strong Rationality (Evans-Guesnerie (1993)).

LOCAL STRONG RATIONALITY: *Let* $\overline{X}$ *be an equilibrium and let* $A(i)$ *and* $A = \int A(i)\, di$ *be the associated matrices when the above axiom is written in* $\overline{X}$. *Assume that* A *is semi-simple. Let* $\mathcal{B}$ *be the "eigenvectors" basis of* A *and let* $n(i)$ *be the norm of* $A(i)$, *induced by the Euclidean norm on* $\mathcal{B}$.

If $\int n(i)\, di < 1$, *the equilibrium is Locally Strongly Rational.*

For a proof, the reader is referred to the original Evans-Guesnerie paper. A serious discussion of the statement is however in order.

General Comments

(i) The condition $\int n(i)\, di < 1$ is a *sufficient condition* for SR. It is necessary and sufficient in special cases such as the one-dimensional case analyzed below.[31] It is however not necessary in general: for example, if matrices $A(i)$ have the same basis but different eigenvectors associated with the eigenvalues of highest modulus, then the reader will check that the above condition is strictly too demanding for SR.[32] Finding necessary and sufficient conditions in relevant subclasses of problems is an open question.

[31] Or in the case, also evoked later, of "essentially identical" agents or more generally of identical eigenvectors associated with the eigenvalue of highest modulus.

[32] Take the case $A = \frac{1}{2}A_1 + \frac{1}{2}A_2$; assume that the eigenvectors are similar, B_1 and B_2, with eigenvalues λ and 0 for A_1 and 0 and λ for A_2. When A transforms any vector x into $(\lambda/2)x$, $(\frac{1}{2}n(1) + \frac{1}{2}n(2))$ is λ. Hence IE-Stability obtains for $\lambda \leq 2$, when our condition holds for $\lambda \leq 1$ and one checks that a "ball of beliefs" is contracted whenever $\lambda \leq \sqrt{2}$ (although SR obtains for $\lambda \leq 2$).

(ii) If $\|.\|$ designates any matrix norm, then, from (an adaptation of) a standard result,

$$\left\| \int A(i)\, di \right\| \leq \int \|A(i)\|\, di.$$

As the left-hand side matrix is A, if the norm under consideration is the norm induced by the Euclidean norm on $\mathscr{B}$, then the left-hand side equals the highest modulus of an eigenvalue of A. And, the right-hand side is nothing else than $\int n(i)\, di$. Indeed, if $\int n(i)\, di \leq 1$, $\|A\|$ is also smaller than one: *Strong Rationality implies IE-Stability.*

In other words, the fact that $\|A\| \leq \int n(i)\, di$ reflects that SR is more demanding than IE-Stability; the discrepancy between $\int n(i)\, di$ and $\|A\|$ may be viewed as an upper bound on the extent to which heterogeneity of expectations is an obstacle to coordination (see footnote 32).

Simple Special Cases:

(i) The next remark is the following: Assume that agents are essentially identical in the sense that their individual $A(i)$ matrices are proportional: $A(i) = \lambda(i)A$, with $\lambda(i) \geq 0$. Then, $\int \|A(i)\|\, di = \int \|A\| |\lambda(i)|\, di = \|A\| \int \lambda(i)\, di = \|A\|$: SR is equivalent to IE-Stability. Hence, what matters is not the heterogeneity of expectations per se, but the interplay between heterogeneity of expectations and the heterogeneity of individual "reactions" to expectations, as reflected in the differences of structure of the matrices $A(i)$.

(ii) The role of the heterogeneity of expectations is particularly easy to analyze in a one-dimensional world. There $A(i)$ is a number, let us write it $a(i)$, and $\|A(i)\| = |a(i)|$. A is also number a: $a = \int a(i)\, di$. Then, if all $a(i)$ have the same sign, let us say positive, the above expression $\int n(i)\, di$ equals $\int a(i)\, di$, which equals a, which equals $\|A\|$. Again, and indeed *agents are still "essentially identical,"* (*local*) *IE-Stability coincides with Strong Rationality.*

The same is true if all $a(i)$ are negative. However, if some $a(i)$ are positive, let us say for $i \in I_1$, and some others are negative, let us say for $i \in I_2$, then $a = \int a(i)\, di$ is, let us say positive, and strictly smaller than $\int n(i)\, di = \int |a(i)|\, di = \int_{I_1} a(i)\, di + \int_{I_2} |a(i)|\, di$. Hence Strong Rationality is strictly more demanding than IE-Stability.

The reason is intuitively easy to understand: by allowing heterogeneous expectations, the analysis of Strong Rationality leads us to consider situations where agents in I_1 have, for example, "optimistic" beliefs, let us say $\epsilon \geq 0$ above equilibrium, when agents on I_2 are on the contrary, "pessimistic," let us say ϵ below equilibrium: the effect on the system is $\epsilon \int |a(i)|\, di$ instead of ϵa if beliefs are homogeneous and universally optimistic.

Indeed, in the case where small reactions of a system to a change of homogenous expectations would result from the combination of large reactions of opposite sign, then the discrepancy between SR and IE-Stability might be considerable.

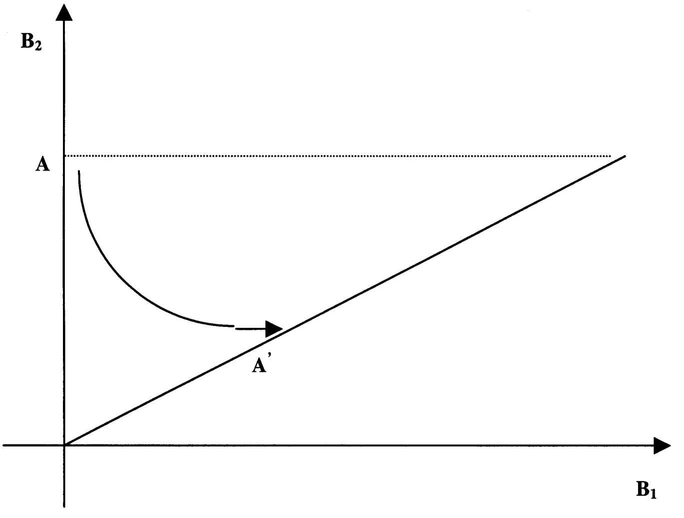


FIGURE 2

(iii) The spirit of the one-dimensional analysis would extend to the case where all matrices $A(i)$ have the same basis (that has then to coincide with $\mathcal{B}$, the basis of A) and the same eigenvector associated with the eigenvalue of highest modulus. Hence, $n(i)$ is the highest modulus of an eigenvalue of the matrix $A(i)$ and $\int n(i)\,di$ is the sum of such moduli.

More on the Heterogeneity Problem

In the case where the matrices $A(i)$ have different bases, then the number $n(i)$ is larger than the modulus of the eigenvalue of highest modulus. Take the case suggested by Figure 2. Then $\mathcal{B} = (B_1, B_2)$ is orthonormal and some $A(i)$ has eigenvectors that are extremely close, one of which coincides with the (horizontal) eigenvector of $\mathcal{B}$. Take the vector O along the vertical axis (of norm 1 in B): it has a very large component along the second (nonhorizontal) axis of $A(i)$. Assume that the eigenvalue of $A(i)$ associated with this axis is ϵ, when the other one is (close to) zero; then the transform of $O\Delta$ is close to $O\Delta'$, a vector whose Euclidean norm in $\mathcal{B}$ is (possibly much) greater than ϵ. In other words, the norm of $A(i)$ induced by the Euclidean norm on $\mathcal{B}$ is much larger than ϵ. In expectational terms, a change of expectations of $M \cdot i$ along $O\Delta$, that would contribute, if all agents had the same pattern of influence on the system as $M \cdot i$ has, to a global reaction that is in some sense "contracting," may trigger an amplification effect, if agents have different patterns of influence, as just suggested.

4.2. *General Intuition from the Analysis*

As announced and argued previously, one of the main objectives of the analysis is to develop simple and economically appealing intuitions about the factors that favor coordination. Here, one will state two rules that are referred to as GI (General Intuition) 1 and GI2. GI1, the key insight that lies behind the above results concerning IE-Stability ($\|A\| \leq 1$) or Strong Rationality ($\int n(i)\, di \leq 1$) is desperately simple!

GI1: *Coordination is favored whenever agents' actions are not too responsive to expectations*.

Note however that this somewhat too obvious intuitive precept helps appraising but does not exhaust the understanding of the specific factors that govern expectational coordination in specific problems. For example, in the Muthian model already sketched, both low supply elasticity and high demand elasticity (that in some sense, everything equal, "stabilize" the equilibrium price) contribute to reduce the responsiveness of actions to expectations. In a simple three-goods neo-Keynesian model where firms have to decide simultaneously about hiring of workers with fixed wage, then besides the already mentioned "Muthian" factors, a high Keynesian multiplier, that alleviates strategic substitutabilities, also favors expectational coordination; and these effects remain effective when, with flexible wages, the setting becomes Walrasian (Guesnerie (2001)).

Indeed, GI1 appears under a large variety of forms in previous studies that put emphasis on "eductive" coordination in models entering the present, or a close enough, setting. This includes Industrial Organization models such as the celebrated Cournot model, where Gabay-Moulin (1980) studied "dominance solvability" of the equilibrium,[33] and the Bertrand model where prudent responses to changes in the pricing expectations of others explain coordination (Borgers (1992), Janssen (1991)). This includes models in Finance where Guesnerie-Rochet (1993) stress destabilizing effects of the opening of future markets. This finally include macroeconomic models, such as the one in Bryant (1983), which stressed the role of strategic complementarities in the process of expectational coordination.

Indeed, a literature on games with strategic complementarities, with a number of applications in Industrial Organization, has developed analysis partly related to the present one (Cooper-John (1981), Milgrom-Roberts (1998), Vives (1990), Milgrom-Shannon (1994)).

Although starting from different preoccupations, this literature has derived a number of sufficient conditions for the Strong Rationality of equilibria (with our terminology) that have counterparts in our setting.

One conclusion however, associated with GI1, is that, so far as the local aspects of coordination problems are concerned,[34] it is not so much strategic complementarities or strategic substitutabilities per se that are instrumental: it is the

[33] This is, to the best of my knowledge, the first model (although it does not enter the present framework) where the idea of GI1 appeared at least implicitly. The Cournot equilibrium is Strongly Rational, with the present terminology, whenever the duopolists actions are not too reactive to expectations.

[34] It is not meant that the present model allows derivation of and encompasses most results of the strategic complementarity literature! The first obvious reason is that this literature considers

magnitude of the response of actions to expectations that matter, not so much their sign (at least when the sign is constant across agents). Also, the sensitivity of actions to expectations is "destabilizing," even when the considered expectations concern other sectors producing substitutes (Guesnerie (1992)).

The second lesson of this analysis is that heterogeneity does matter (again a criticism of the representative agent)! It is less loosely stated as GI2.

GI2: *The presence of many heterogenous agents perturbs coordination, insofar as the responsiveness of the system to expectational changes of given magnitude is generally exacerbated when both individual reactions and expectations become more heterogeneous.*

The exact role of heterogeneity in the perturbation of coordination is pretty clear in the one-dimensional case and reasonably clear in the simple cases that have been evoked. In other circumstances, the interplay of the heterogeneity of agents with the dispersion of expectations may be difficult to assess and the present theory needs elaboration to help to improve intuition.

5. COORDINATION WITH SEQUENTIAL DECISIONS: A BRIEF APPRAISAL

As already stressed, the argument of the previous section rests on an extreme timing assumption, i.e. that decisions are made simultaneously or, equivalently for our purpose, that no decision-making unit knows, at the time he or she decides, the decisions taken by the others.

In this section, one will sketch how the analysis is modified when decisions cease to be fully simultaneous. More precisely, suppose that the unique decision node of the preceding model ($t = 1$) is replaced by T decision nodes ($t = 1, \dots .T$). Hence, the population is divided into T different groups, each assigned one decision node: at any time t however, all previous relevant aggregates are public and CK. We assume, for the sake of simplicity, that the T groups under consideration have the same size and are identical, i.e. that each is a *representative sample* of the whole population of agents.[35]

In this setting, the static equilibrium of the previous section obtains as an intertemporal equilibrium. For example, in the case $T = 2$, under the representative sample assumption, the reader will convince himself that this equilibrium

general games where agents do not react, as here, to some aggregate of the others' actions but to individual strategies; this framework is "essentially" more general than the present one. Also, this literature stresses global results that we cannot replicate in our setting: the strategic complementarity hypothesis, particularly when coupled with uniqueness, plays a role for obtaining *global* dominance solvability. Such a "coupling" effect does not hold with strategic substitutability. Hence, there is a sense in which strategic complementarities make coordination easier (see, however, footnote 38).

[35] The representative sample assumption simplifies notation and statements rather than affects the essence of the argument.

consists of the aggregate $(\bar{X})/2$ in the first period and $(\bar{X})/2$ in the second period. The following can be shown:

Local Strong Rationality in a Sequential Setting, $T = 2$: *In the sequential setting just described, whatever the matrix A of the simultaneous previous version of the problem:*

(i) $\int n(i)\,di < 2$ *is always sufficient for LSR of the second period equilibrium, where $n(i)$ are defined as previously;*

(ii) $\int n(i)\,di < 1$ *is always a sufficient condition for LSR of the sequential equilibrium.*

Furthermore, if all eigenvalues of A have a negative real part, $\int n(i)\,di < 2$ is sufficient for LSR.

This statement shows that, in the sequential setting under consideration, the conditions for LSR are often weakened when compared to the simultaneous decision case.[36]

The (sketch of) proof I shall give (together with the assumptions that are required) is deferred to the Appendix.

The main message associated with the previous statement is clearcut: although the intertemporal equilibrium outcome is unchanged, coordination is generally becoming easier when the decisions are made sequentially rather than simultaneously. More precisely, the fact that the second period equilibrium is easier to guess than the simultaneous equilibrium fits a most intuitively plausible remark: it is easier to observe what the others do rather than to predict![37] The fact that coordination is also easier at the first period obtains when the first period outcome and the second period outcome have some substitutability, in the precise sense imbedded in the last part of the statement (e.g., if $-A$ is an M-matrix): the property is intuitive in the one-dimensional case, where a negative real part involves a negative eigenvalue, implying that an increase in x_1 involves a decrease in the equilibrium reaction x_2. Note that this latter property occurs in the Muthian model so that the present statement does imply the previous conclusion obtained in this setting (Guesnerie (1992)).[38]

[36] This setting allows a large variety of cases to be encompassed going from the case where decisions are simultaneous to the case where, when T is large, they are (almost) fully sequential.

One would like the following to be true:

$\int n(i)\,di \leq T - \lambda(T)$ *where* $\lambda(T)/T \to 0$ *when* $T \to +\infty$ *is sufficient for LSR.*

If so, conditions for LSR become weaker and weaker when T increases. In particular, in a "given" problem, the condition becomes satisfied from a "large enough" T: there is, hence, some kind of monotonic connection between the "fully" sequential case (which, roughly speaking, in this model requires $T = \infty$) where Strong Rationality results from "backwards induction" arguments, and the "fully" simultaneous case. I conjecture that the statement holds in a large variety of situations, including the last case considered in the statement.

[37] Note, however such an obvious fact does not necessarily enter easily other learning theories!

[38] It was argued in a previous footnote that there was a sense in which strategic complementarity facilitated coordination; there is also a sense, in this sequential setting, in which it makes coordination more difficult.

The general intuition that I choose to stress from the above analysis first refers to the looser lesson drawn from the examination of the second period coordination problem, and then to intertemporal coordination:

GI3: *Partial observation of the others' actions alleviates the difficulties of* (*collective*) *guessing so that*, *in general*, *coordination is easier when decisions are made sequentially at exogeneously fixed nodes*, *particularly when equilibrium decisions tomorrow are in some sense "substitutes" for observed decisions today.*

The reader may make two remarks.

(i) First, although the systematic result that is suggested here describes mental processes that exhaust the infinite sequence of elimination of non-best responses, it is left to the reader to convince himself that the essence of GI3 would be preserved if, accepting less perfect coordination, we had focused on processes involving only a few rounds of mental activity. However, it is true that when the number of subperiods under consideration substantially increases, the thought process of an agent acting in the first period must envisage all the following periods, which, everything equal concerning the depth of mental processes at every period, increases the complexity of the guessing game.

(ii) A clear limitation to the argument made here and to the validity of the intuition that it captures is the exogeneity of the timing of decisions. An endogenous timing of decisions[39] would introduce a new and different dimension to the problem under consideration here. Note however, without going further into that question, that in models such as those just evoked where timing coordination is governed by complementarities or substitutability considerations, the general views on eductive stability developed earlier are likely to remain relevant.

PART III: Coordination, Uncertainty, Information, and Time

The analysis of the previous part has been developed in models that are deterministic or more accurately where uncertainty was solely uncertainty on the actions of the other agents and hence might be called "extrinsic."[40] The present part of the paper attempts to review the coordination problem, within the "Common Knowledge" methodology under scrutiny here, in settings where there is "intrinsic" uncertainty that affects the fundamentals of the economy.

Also, most of the analysis of the previous section considers a quasi-static, or in another interpretation, short-sighted model, whereas this part also concentrates attention on longer horizon, in fact on infinite horizon, models.

Concerning intrinsic uncertainty, different directions of reflections are open. Let us mention three of them.

[39] Such as considered by Chamley-Gale (1994).

[40] I use here the terminology rather than the concepts coming from Cass-Shell (1983): uncertainty is "intrinsic" whenever it concerns the "fundamentals" of the economy. The remaining uncertainty bears on the actions taken by the agents that, even if coordinated in a rational expectations manner, may be triggered by exogenous sunspot events. Such a behavioral uncertainty may be called "extrinsic."

A first obvious suggestion is to assume that, in the type of models previously considered, the outcome is "noisy" rather than deterministic.

A second suggestion directs us to a recognized chapter of the literature, the concern of which is the learning of a relevant economic parameter, unknown at the outset but progressively revealed through time.

A third direction of reflection brings us back to another known chapter of the literature, concerned with the case when information on a relevant parameter is dispersed within the society but may be, in its entirety or in part, transmitted through public economic variables like prices.

Indeed, the first and second directions suggested will be both discussed in the next section entitled "Progressive Learning with Symmetric Information." The third item will be considered in the section entitled: "Learning Information through Prices."

The third section switches attention from information to time and envisages expectational coordination from the CK methodology viewpoint in infinite horizon models.

It should be stressed at the outset, that contrary to the previous Part, which has exposed somewhat mature theory, the present Part examines more exploratory reflections.

6. PROGRESSIVE LEARNING WITH SYMMETRIC INFORMATION

As made clear in the title, this section considers a polar case of information transmission, where the agents initially have symmetric information on a parameter and where new information that arises when time passes is public so that information always remains symmetric.

6.1. *Model and Questions*

Again, the argument will essentially take place in the framework of (an intertemporal variant of) the Muth model, but in an intertemporal variant where supply and demand curves are time-invariant and affected by noise and where inventories are ruled out.

More precisely, the demand and supply curves at period t are assumed, partly for the sake of simplicity, to be linear and are written as follows.

DEFINITION 3 (*Demand*): $D(p_t) = A - Bp_t + \widetilde{\mu}_t + \theta$, where A and B are time-invariant parameters, $\widetilde{\mu}_t$ is an i.i.d random shock, and θ is an unknown parameter that has been drawn at the outset by Nature according to a probability distribution that is public knowledge.

DEFINITION 4 (*Supply*): $S(p_t) = Cp_t$.

The timing of the problem is now without mystery: at time $t-1$, farmers make decisions on their crops, which, as stressed above, are entirely sold at time t, and the story starts again.

While avoiding a formal definition, let us make clear what the intertemporal rational expectations (Nash-Bayesian) equilibrium of this repeated version of the farmers' problem is.

The price at each period is a random variable $\widetilde{p}_t^*$, whose value is determined as follows:

$$A - Bp_t^* + \mu_t + \theta = CE(p_t^*/t-1)$$

where μ_t is the realization of the random variable $\widetilde{\mu}_t$, and where $E(p_t^*/t-1)$ is the rational expectation at $t-1$ of the equilibrium price at t.

Taking expectations (at time $t-1$) on both sides, we can derive the rational expectations $E(p_t^*/t-1)$ as a function of the exogenous parameters and of beliefs on θ. The latter have been updated as follows: starting from period 0, where $E(\theta)$ is evaluated from the initial prior, the observation of p_1 at period 1, together with the knowledge of $E(p_1^*/0)$, allows to ascertain the realization of $\mu_1 + \theta$. The distribution of θ is then continuously updated, using Bayesian principles, at period $2, \ldots, t-1$, and so on. We have then completed the informal description of the intertemporal equilibrium.

Now, in this simple model, where information always remains symmetric, four separate but related questions can be raised:

Question 1: Forgetting for the moment about "intrinsic" learning (which here means—see footnote 40—learning fundamentals), i.e., focusing on the case where θ is known at the outset, is "extrinsic" learning (learning the other's actions—again see footnote 40) alone successful? Or is the intertemporal equilibrium, in this case, Strongly Rational?

Question 2: Is it the case that, when θ is unknown and progressively learned through Bayesian updating, the intertemporal equilibrium is Strongly Rational?

In our setting, the answers to Questions 1 and 2 obtain from the following result on the "Strong Rationality" of the one-period equilibrium (Guesnerie (1992)).

Whatever the precise characteristics of the perturbing noise, the equilibrium of the static model is Strongly Rational,[41] *for adequate CK restrictions, if and only if the elasticity condition of Section 2 is satisfied, i.e., if and only if $B/C \leq 1$.*

In other words, the conditions for SR are approximately the same in the noisy and in the non-noisy models of Section 2.[42]

[41] Note that here, due to the linearity of the system, both the initial restriction, and, hence, Strong Rationality are global.

[42] Let us say a few more words on the more general question of what happens to the analysis of the previous section when noise is added. The bottom line is the following. Assume that in the model of the previous section, the outcome is affected by some intrinsic noise (so that for given actions of the agents, the outcome is not a deterministic vector X, but a random one). The factors that govern eductive stability and favor "Strong Rationality" seem to be very similar to those that have the same role in the non-noisy model. With appropriately adapted Common Knowledge restrictions, the proposition on p. 453 will have counterparts. Such counterparts may however be more or less close, in particular depending on whether the system is linear or, on the contrary, nonlinear and in the latter case whether the noise is small or large. Existing statements along these lines, in the

The reader will convince himself that in the special separable setting that we have here, SR in the intertemporal equilibrium is a consequence of SR in the static (one-period) version, so that, under adapted CK restrictions, we can state the following:

The elasticity condition $B/C \leq 1$, is sufficient for the Strong Rationality of the intertemporal equilibrium.

The next question separates intrinsic from extrinsic learning (assuming that it has been successful).

Question 3: Is "intrinsic" learning, taken in isolation, successful? Precisely, is it the case that in an intertemporal Nash-Bayesian equilibrium, the updated beliefs of all agents converge to the true value?

Note that this last question is the one mainly treated by a full segment of the literature that takes for granted, contrary to what we do here, that the coordination of actions, even if it is affected by the learning problem, obtains in a Rational Expectations-like manner, i.e. within an intertemporal Nash-Bayesian equilibrium, with Bayesian updating of beliefs. Will, along an equilibrium path, the true value of θ be asymptotically learnt?[43] Indeed, powerful tools (Martingale Convergence Theorems) (see Bray and Kreps (1988)) apply and can be used to prove that in many contexts asymptotic learning does take place. In particular, here we have the following:

In the intertemporal equilibrium, under weak (*probabilistic*) *conditions, the estimation of θ will converge to the true value.*

Question 4: Are both types of learning together successful?

A positive answer to Question 4 implies that the intertemporal equilibrium is both asymptotically revealing and Strongly Rational, i.e. that "extrinsic" ("eductive") learning (learning the others' actions) and "intrinsic" ("evolutive," learning θ) learning are together successful. It indeed obtains the following.

Under the above elasticity condition, the intertemporal equilibrium is Strongly Rational and θ is asymptotically revealed, so that, in our setting θ is CK in the long run.

The picture, at this stage, seems rather clear. Its generality, however, deserves some discussion. Let us say, in a nutshell, that it seems true that a positive answer

---

one-dimensional version of the above model, can be found in Guesnerie (1992) (as just recalled) and Guesnerie-Rochet (1993). One insight of Guesnerie (1992), not evoked later, is that in the linear version of the Muthian model, uncertainty is likely to accelerate the speed of convergence of the mental process. Also, the effect of risk-aversion on convergence of the mental process is ascertained in complementary contexts in the two papers just mentioned. The provisional conclusion, which will be stressed in an informal way later through GI4, is however that the effects of noise on "eductive" coordination are ambiguous.

[43] The question has generated many studies that originate (at least) in Townsend (1978). A very limited sample includes Bray (1982), Fourgeaud-Gourieroux-Pradel (1986), Bray-Kreps (1988), Vives (1993). These contributions focus attention either on this setting or on variants that are more complex in some respect (for example, in Vives (1993), the noise is an ARMA process, and information is split in the society and hence not symmetric.)

As it has been noted in the literature, the speed of convergence of the process is a question that may be more important than convergence itself. We are however not concerned with this question here.

to Question 4 requires a positive answer to question 1[44] but the equivalence that holds here between Questions 1 and 2 is more doubtful.[45]

It remains that the existing positive results of the literature on intrinsic learning suggest that the binding constraint, for a positive answer to question 4, is the success of "extrinsic learning."

I propose to formulate all these teachings in an informal and rather prudent way, under the form of a fourth and a fifth general intuition.

GI4: *The addition of "intrinsic" exogenous noise in the systems under consideration in the previous part may either improve or hinder coordination*.

GI5: *The asymptotic learning success of Bayesian agents having symmetric information on an unknown parameter of the system is primarily constrained by the success of "eductive" learning*.

7. LEARNING INFORMATION THROUGH PRICES

In the other polar model under consideration now, there is one relevant parameter on which information is a priori privately held and dispersed in the society in an asymmetric way. In influential papers of the end of the seventies (Grossman (1976), Radner (1979)), it has been argued that in a number of circumstances, fully revealing equilibria[46] would emerge: in such equilibria, all the socially relevant amount of information would be reflected in prices, and hence accessible costlessly to everybody.

The question that is explored in this section is whether such price-revealing equilibria can be "anchored in CK." Are they likely to be educed through the mental processes associated with the Strong Rationality construct?

[44] The reason is very simple in this model and looks fairly general. Let us consider the continuation equilibrium starting far away in time, once θ is approximately known; then, this continuation equilibrium, which has to be SR, is very close to the deterministic intertemporal equilibrium envisaged in Question 1 and continuity arguments justify the above assertion.

[45] To look at the problem of equivalence between 1 and 2, one can first ask oneself whether a positive answer to Question 2 implies a positive answer to Question 1, i.e. in the present setting whether the fact that eductive extrinsic learning is successful in the random case implies that it will be successful for the deterministic case. The game-theoretical literature describes cases where kind of a "contagion" effect makes eductive learning easier in a context where the data are random rather than deterministic (see Carlsson-Van Damme (1993)), a fact that suggests that the answer to the above question may be sometimes negative. The answer to the converse question: "Does a positive answer to 1 implies a positive answer to 2?" is more likely to be positive. When this is the case, as here, Question 1 is the key question.

[46] Such equilibria were indeed denominated Rational Expectations Equilibria, and it is still the case that in some peoples' minds the idea of Rational Expectations is associated with the information-revealing prices literature. Naturally, it is true that, for example, fully revealing equilibria are Rational Expectations equilibria in the general sense of the word taken in this paper, but the identification of the information-revealing prices literature with the Rational Expectations literature is completely misleading.

7.1. *Model and Information-Revealing Equilibria*

The exploration will be undertaken within the framework of a very simple model. Again, think of farmers. But they are not the farmers of the previous sections who had to choose at every period the volume of the crop in a way that could not be made conditional on prices (since these will be known only after the crop is reaped). Our farmers here can make their order today conditional on a price today.[47] Think of this order as a volume of seeds, or fertilizers, or any input that is required for the production of wheat. Say that each farmer may have some information on some relevant parameter (it might be, for example, the price of wheat tomorrow).

DEFINITION 5 (*Demand*): For every agent, today's (input) excess demand, as a function of the market price today, denoted p, is of the simple form: $E(\theta) - p$, where $E(\theta)$ is the expectation of the relevant and unknown parameter to which we alluded above.

Now, *information on* θ obtains as follows: there are n groups of farmers; in each group, all farmers receive one identical signal, which will however generally differ across groups, and which is correlated with the true value of the parameter. As usual in this literature, it is assumed that there is no communication between agents (or in a weaker way, between groups).[48]

The initial literature on information-revealing price equilibria was focusing attention on an equilibrium concept, the so-called Rational Expectations Equilibria, that associated beliefs with prices but that left in the shadow any problem of "implementation" of the equilibrium. Our purpose here is to examine whether the equilibrium can obtain as the outcome of a process of guessing and second guessing. . . . We aim then at a particular kind of implementation story that cannot be told if the "game-form" of the problem is not fully specified. The specification chosen here is now described:

1. First, the *strategies* used by the agents: These strategies consist of demand curves; each farmer transmits to the "market," which is supposed to be managed by some "auctioneer," a demand curve that expresses demand as a function of prices.

2. Second, the *rules of the game*, announced and followed by the auctioneer: The auctioneer aggregates individual demand curves in order to get an excess demand function. Three cases occur:

(i) If there is only one price equilibrium, then each agent gets what he has demanded at this price.

(ii) If there are several price equilibria, then the auctioneer selects the smallest one and proceeds as above.

(iii) If there is no equilibrium price, then some announced fixed rule is implemented (such as no exchange, use of a given rationing scheme, . . .).

[47] A "limit order" in the theory of finance is an intermediate possibility.

[48] Note that as each farmer is assumed, as previously, infinitesimal, price-taking behavior obtains and coexists with noninfinitesimal private information.

Here is the framework of the model that will be still further specified:

(a) $n = 2$; there are only two groups of agents.

(b) The first group of agents receives one of two possible signals that consist of two numbers denoted R (also associated with a color, here Red) or G (Green) and such that $E(\theta) = R$ or G (we assume $R \geq G$ and note $\Delta = R - G$); the second group of agents receive no signal at all. The first group of agents, which consists of $\alpha\%$, $\alpha > 0$, of the population, is called informed; the second group $((1-\alpha)\%)$ is called uninformed.

We can now define a *fully-revealing demand equilibrium*, where strategies consist of (excess) demand functions.

Definition 6: A *fully-revealing demand equilibrium* consists of the following strategies:

Informed agents submit the excess demand curve

$$R - p \text{ (resp. } G - p\text{)}, \quad \text{if} \quad R \text{ (resp. } G\text{)}.$$

Uninformed agents submit the excess demand curve $d(p)$:

$$d(p) = G - p, \quad \text{if} \quad p \leq M, \quad \text{with} \quad M \in \left[G + \frac{1-\alpha}{2-\alpha}\Delta, G + \frac{\Delta}{2-\alpha}\right],$$
$$d(p) = R - p, \quad \text{if} \quad p \geq M.$$

Given these strategies, and the auctioneer selection rule, the equilibrium price is $p = G$ if G, $p = R$, i.e., it is fully revealing, in the standard sense of the term.

The reader will convince himself that excess demands conditional on both signals are on Figures 3a and 3b. The result follows.

Hence, in the simple model under consideration, a fully-revealing equilibrium exists. It is left to the reader to show that no other equilibrium can exist.[49] Is the unique equilibrium Strongly Rational, in our terminology?

7.2. "*Educing*" *Information-Revealing Equilibria*

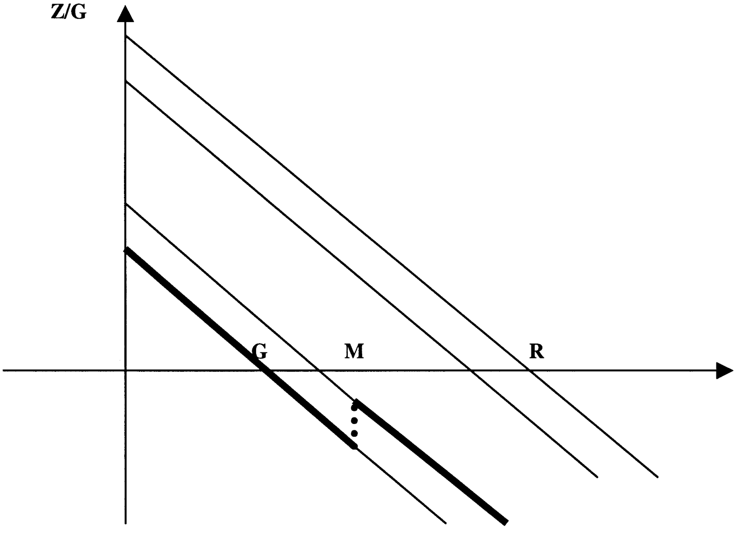
Whenever the proportion of informed agents is large enough, a two-round mental process is enough to "educe" the equilibrium. The argument proceeds as follows:

Round 1: For informed agents, "playing" $R - p$ (resp. $G - p$), if R (resp. if G), is a dominant strategy. Uninformed agents discard strategies d that do not satisfy $G - p \leq d(p) \leq R - p$.

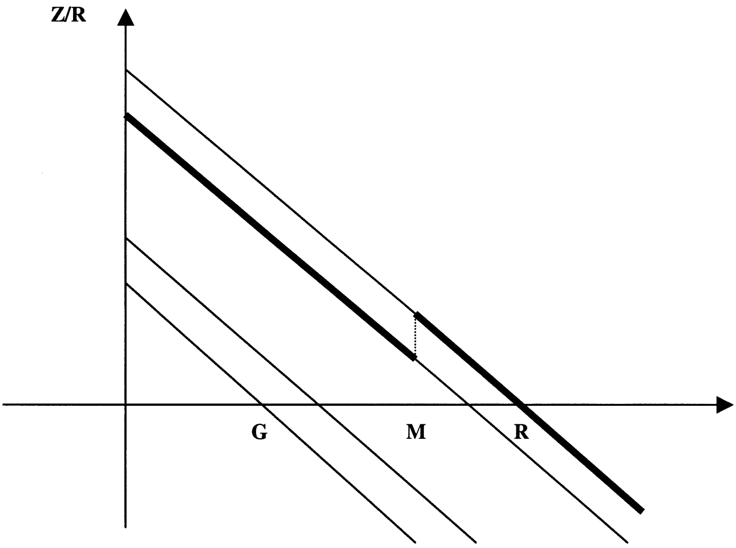
From that, it follows, that the equilibrium price can only be either in $[G, p_G]$, if G, or in $[p_R, R]$, if R, where $\alpha(G - p_G) + (1-\alpha)(R - p_G) = 0$ and $\alpha(R - p_R) + (1-\alpha)(G - p_R) = 0$. If α is large enough, then $p_G \leq p_R$.

[49] Whatever the beliefs of the noninformed, they transmit demand functions that trigger two different equilibrium prices, if R or G obtains, so that the induced beliefs, which in equilibrium have to coincide with the starting ones, must associate fulfilled point expectations with two different prices that can only be $p = R$, $p = G$.

P. Bossaerts (1997) considers similar equilibria, but with a different rationale for the auctioneer.



(a)



(b)

FIGURE 3

Round 2: Uninformed agents make their demand coincide respectively with $G - p$ and $R - p$ on the two segments just mentioned. This suffices to generate a fully-revealing equilibrium.[50]

[50] It should be clear that contrary to the previous Sections, we here are testing global and not local Strong Rationality (note however, that equilibrium allocations, rather than equilibrium strategies are here unique).

The reader will have noticed that in the present argument excess demand always has a unique zero, so that it is not necessary to appeal to the selection rule. This reflects the fact that the proportion of informed agents is large enough. Indeed when this proportion decreases, the argument is unconclusive in two rounds, but can be pursued using the selection rule. It is shown in Desgranges-Guesnerie (2000):

SR OF REVEALING EQUILIBRIA: *The above equilibrium is Strongly Rational, whatever the proportion of informed people.*

The number of iterations required for the mental process to converge increases when the proportion of informed agents decreases.

The argument extends, with minor amendments, to the case where the number of possible signals is finite and greater than 2.

Hence, the "eductive" stability of the price-revealing equilibria, which is straightforward when the proportion of informed agents is high, relies on a somewhat more sophisticated argument that uses the coordination value of the CK selection rule, when this proportion is smaller. It should be clear however that the case we are considering is rather special in one respect: the information held by each informed agent equals the total information of the Society; in that sense information can be called "sharp" or even very "sharp." It is intuitively clear that this property has played a positive role in the argument: the behavior of informed agents does not depend on their beliefs but only on the signals they have received; consequently, uninformed agents make rather sure inferences on the plausibility of prices as a function of information.

If there were two groups of informed agents receiving different signals, then the behavior of the informed agents would necessarily be more hesitant (no dominant strategy any longer) and the mental process less firmly started. Going to the other extreme, if there were a large number of groups of informed agents, each group receiving a (possibly) different signal, then each agent would "trust" the market more than its own signal, a fact that is likely to paralyze the mental process. "Diffuse" information, as opposed to "sharp," is likely to be detrimental to the convergence of the mental "eductive" process. Where does information cease to be "sharp"?

Desgranges (1999) provides an answer in the context of the present model, when the (finite) set of signals has no a priori structure: roughly speaking, "sharpness"[51] appears as a necessary and sufficient condition for the following three properties: existence, uniqueness, and Strong Rationality.

We leave it to the reader to convince himself that the information is not "sharp," but rather "diffuse," in the example of Figure 4.

In this example, there are four groups, an agent in each group either receives R or V, so that the state of the world can be identified with a four-letter (number)

[51] The formal concept of "sharpness" that is considered can be, roughly speaking, defined as follows: given any couple of states of the world, at least one agent knows that the second has not occurred when the first one occurs.

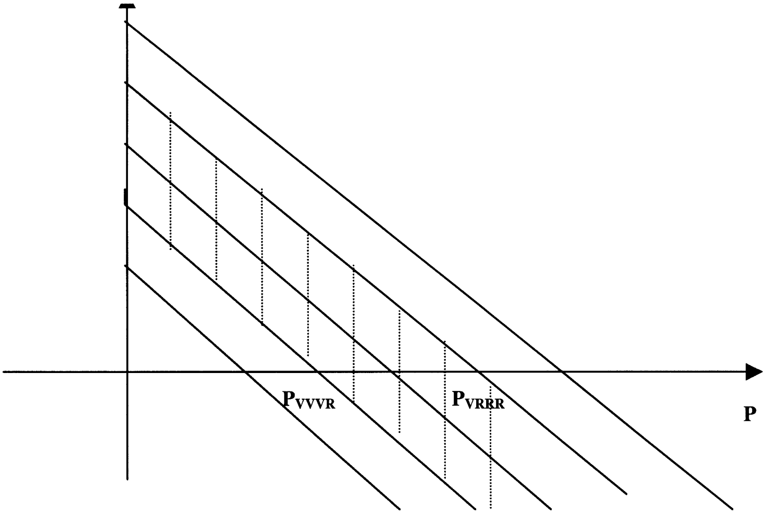


FIGURE 4

sequence such as *RVVR*. In the example, we choose to identify θ with the sum of the numbers, so that *RVVR* and *VRVR* transmit identical optimism on the value of the relevant random variable. The set of rationalizable prices consists of the segment p_{VVVR}, p_{RRRV}: the mental process stops far away from the revealing equilibrium.[52]

The lessons of this brief investigation can be summarized in one "general intuition," which at this stage is admittedly fragile:

GI6: *Coordination on price-revealing equilibria, and hence instantaneous learning of information, is more likely when the information of informed agents is "sharp" rather than "diffuse," and when the proportion of informed agents having sharp information is high.*

Clearly, work remains to be done to test the general validity of the intuition as well as its usefulness.[53]

[52] As noticed above, the difficulty has the following intuitive flavor: agents trust too little their own information. This phenomenon is reminiscent of the one that slows down the convergence of real time evolutive processes as assessed in Vives (1993). Because, as time passes agents have given less and less weight to their own piece of information, their action reflects less and less their information and the aggregate transmits less and less additional information, so that the speed of real time learning decreases, in such a way that convergence is in $1/\sqrt{t}$.

[53] I voluntarily stick to the analysis and formulation of the original lecture. On the one hand, the lessons of the elementary model have to be qualified (Desgranges, Geoffard, Guesnerie (2000)). On the other hand, the results of Desgranges (1999), Heinemann (2001) confirm the fruitfulness of the "sharpness" idea.

8. INTERTEMPORAL INFINITE HORIZON MODELS

8.1. *A General Principle*

In this section, we consider models that generate temporal paths, that are described by finite dimensional vectors indexed by t, going from 0 to $+\infty$, and denoted y_t.

In order to test whether a Rational Expectations Equilibrium path $E^* = \{y_t^*\}$ is Locally Strongly Rational, we shall refer to two different CK restrictions.

The first one is simply the neighborhood restriction introduced at the outset of this paper, which is restated for the sake of convenience.

It is CK that E, the trajectory of the system, belongs to $V(E^*)$, a (small) "neighborhood" of E^*.

It must be realized that, as trajectories are infinite-dimensional objects, there is no longer a natural or undisputable notion of neighborhood that should be adopted. The CK assumption has then to refer to an explicit topology. Given *the intertemporal equilibrium* under scrutiny, the fact, stressed in Section 2, that it *cannot be LSR without being determinate* is no longer trivial: note, however, that it is true only if the topology in the definition of the restriction and the topology that serves to test determinacy are the same!

8.2. *Models and CK Restriction*

From now on, attention shall be concentrated on two categories of models.

8.2.1. *One-Dimensional, One-Step-Forward Looking Models*

The first one consists of models, whose prototype is the simplified OLG model, in which the state variable is essentially one-dimensional and equilibrium today y_t[54] is determined from the expectations of equilibrium tomorrow. Formally, when expectations are deterministic point expectations y_{t+1}^e, the reduced form of the system is

$$y_t = \varphi(y_{t+1}^e).$$

When expectations are stochastic but homogenous, we have

$$y_t = \Phi(\mu_{y_{t+1}})$$

where $\mu_{y_{t+1}}$ is a probability distribution describing the (homogenous across agents) expectations over the possible equilibrium values tomorrow.

[54] In view of the general notation used above, the state of the system, E in Section 2, or X in Section 3, consists now of the infinite sequence $(y_1, \ldots, y_t, \ldots)$.

The present subsection relies much on Guesnerie (1993); related literature includes Bryant (1984), De Castro (1997), and Heinemann (1997). For views on the models considered here, see Azariadis (1993). For overviews on intertemporal learning, see Evans-Honkapohja (1998, 1999).

When expectations are stochastic and heterogenous, then

$$y_t = \Theta(\Pi_i(\mu^i_{y_{t+1}}))$$

where $\mu^i_{y_{t+1}}$ is the probability distribution describing expectations of agent i and $\Pi_i(\mu^i)$ denotes the Cartesian product of these measures.

The standard stationary equilibrium concepts: *steady states, cycles, and sunspot equilibria* can easily be defined using the functions previously introduced; this is left to the reader.

Naturally, the functions φ, Φ, Θ are not independent and have connections, some of which are obvious and fully general, some of which are less general and less obvious. The only formal property I shall need here for a sketch of the proof is a first order approximation of Θ around a steady state. Axiom 2 is indeed a special reinterpretation of Axiom 1 (in Section 4), rewritten around the steady state and for the one-dimensional variable y.

AXIOM 2: *Assume that each individual agent i's expectations of the state of the system are described by a random variable* $\tilde{y}(i)$, *whose support is in* $V \subset R$, *a neighborhood of a steady state* y^*, *and whose mean is* $y(i)$; *then*, *if* V *is small enough*, *the state of the system* y' *is*, *to the first order approximation*, *such that*

$$y' - y^* = \int a^*(i)[y(i) - y^*]\, di \quad \textit{where} \quad a^*(i) \in R.$$

The reader may note that the above axiom implies[55] the "Consistency of Derivatives" Axiom that was introduced in the sunspot literature by Guesnerie (1986) and used in particular by Chiappori-Guesnerie (1989) and Chiappori-Geoffard-Guesnerie (1992).

The reader will check, as in Section 4.2, that the axiom involves the following relationship between the derivative of the deterministic function φ and the $a^*(i)$:

$$(\partial\varphi/\partial y)_{y^*} = \int a^*(i)\, di.$$

[55] Assume that beliefs are homogenous and consist of a random variable $\{y_1, \dots, y_n\}$, where the y_l are close to y^*, and are assumed to appear with probability π_l; then,

$$y' - y^* = \int a(i)\left(\sum_l \pi_l[y_l - y^*]\right) di,$$

$$y' - y^* = \left(\int a(i)\, di\right) \sum_l \pi_l(y_l - y^*),$$

so that the derivative of y' with respect to y_l is $\pi_l(\int a(i)\, di) = \pi_l a$. This is the "consistency of derivatives" axiom. This latter axiom, however, does not say anything on heterogeneous expectations and hence cannot imply the present axiom.

8.2.2. *Adding Memory into the Model*

The second type of model is more general than the previous one, by incorporating a memory effect, i.e. a variable y_{t-1}. Indeed, we shall restrict attention to a linearized version of the model, i.e.,

$$y_t = \alpha y^e_{t+1} + \beta y_{t-1},$$

where α, β are two real coefficients.

The previous axiom can however be transposed to systems with memory without formal change, since we are interested in change of the present when the past variable is fixed. This implies that the behavior of the linear system when agent i has random expectations $\widetilde{y_{t+1}}(i)$, with mean $y_{t+1}(i)$, can be formalized as follows:

$$y_t = \int \alpha(i) y_{t+1}(i)\, di + \beta y_{t-1}.$$

($\alpha(i)$ is, here, as $a^*(i)$ of Axiom 2.)

As before, one notes that with homogenous expectations, $y_t = \int \alpha(i) y^e_{t+1}\, di$, so that consistency requires

$$\int \alpha(i)\, di = \alpha.$$

8.2.3. *CK Restriction*

As explained before, CK restrictions have to bear on the trajectories of the system.

Take for example a steady state of the first model,

$$y^* = \varphi(y^*).$$

A most natural stationary CK restriction is as follows.

CK(∞): *It is CK that* $y_t \in [y^* - \epsilon, y^* + \epsilon], \forall t$.

Using the findings of Section 4 that apply when the numbers $a(i)$ are positive, it is intuitively clear that the following set of implications can be (approximately, when ϵ is small) iteratively obtained:

It is CK that $y_t \in [y^* - a\epsilon, y^* + a\epsilon], \forall t$ where $a = \int a^*(i)\, di = (d\varphi/dy)_{(y^*)}$; and

it is CK that $y_t \in [y^* - a^2\epsilon, y^* + a^2\epsilon], \forall t$;

. . . and

it is CK that $y_t \in [y^* - a^n\epsilon, y^* + a^n\epsilon]$;

. . .

It should be noted that the logical status of the CK restriction raises issues that are somewhat more delicate than previously. They are due to the fact, not that agents are in infinite number (this was also the case previously) but that, due to the infinite horizon, some agents may not be born (in the OLG interpretation) or that some may have to act at different periods.

The previous writing of the restriction leans us towards the interpretation of the mental process as a simultaneous virtual process, taking place at the beginning of time, and involving all present and future agents, or towards a reinterpretation of this virtual process as an infinite sequence of mental processes where the CK restriction is maintained throughout and where the process of elimination tomorrow is perfectly anticipated today.

Given the difficulties of interpretation, it may be more satisfactory to view the CK assumption as a time 0 horizon assumption of the following kind:

CK(N): *It is CK at date* $t = 0$, *and it will be CK at subsequent dates* $2, 3, \ldots N-1$ *that*: $y_N \in [y^* - \epsilon, y^* + \epsilon]$, *where* N *is the horizon*.

For example, using the assumption at $t = 0$, and going through a backward induction, the following CK implications obtain, whenever all the numbers $a(i)$ of the Axiom are positive:

$$y_{N-1} \in [y^* - a\epsilon, y^* + a\epsilon],$$
$$y_{N-2} \in [y^* - a^2\epsilon, y^* + a^2\epsilon],$$
$$\ldots$$
$$y_0 \in [y^* - a^N\epsilon, y^* + a^N\epsilon].$$

And under the conditions where the first process converges to y^*, y_0 is very close to y^* when N is large.

This suggests another (equivalent) definition of Local Strong Rationality, to which we will refer in the following:

y_0 *is very close to* y^* *when* N *is large, and the* N*-horizon CK(N) restriction holds.*

8.3. *Strong Rationality in One-Dimensional, One-Step Forward Looking Models*

We successively consider models of the OLG type without memory and then models with memory.

8.3.1. *Models without Memory Effects*

We are in a position to state some formal propositions that summarize (part of) our present knowledge about coordination in the first one-step forward looking model.

ONE-DIMENSIONAL, ONE-STEP FORWARD LOOKING MODEL: *The three following statements are equivalent:*

(i) *The Steady State is "determinate."*

(ii) *There is no Stationary Sunspot Equilibrium in a close enough neighborhood of the Steady State.*

(iii) *"Reasonable" adaptive learning processes are asymptotically stable.*

Furthermore if all the numbers $a(i)$ of the above axiom have the same sign, then a next fourth statement is equivalent to the first three:

(iv) *The Steady State is Locally Strongly Rational.*

Statement (i) is well known to be equivalent to the fact that

$$(d\varphi/dy)_{(y^*)} = a^* \left(\text{here } = \int a^*(i)\, di\right) \leq 1.$$

The equivalence between this condition and (ii) is also well known, and in the present setting follows, for example, from Chiappori-Guesnerie (1989).

There are several definitions of "reasonable" that make (iii) true. In particular, if adaptive learning rules detect cycles of order two, in the sense of Grandmont-Laroque (1986), then assertion (iii) obtains from the application of the conclusions of Theorem 1, in Guesnerie-Woodford (1991).

Finally, the proof of the assertion associated with (iv) has been just sketched above.

The statement just emphasized provides a limit case in which various criteria of expectational stability of different inspiration lead to very similar views of coordination: determinacy, on the one hand, i.e. absence of perfect foresight equilibria in a neighborhood of the steady state; absence of stationary sunspot equilibria in a neighborhood of the steady state, on the other hand, are circumstances that a priori favor coordination. However, the fact that such circumstances are those that also determine local asymptotic stability of a significant class of learning rules or that trigger local Strong Rationality (which here encompasses a natural version of Iterative Expectational Stability) is surprising and cannot be expected to hold generally. Reminder: LSR $\Longrightarrow$ Determinacy but the converse is not generally true.

Let us now say a few words on the "eductive" stability of stationary equilibria that are not steady states, but are, for example, Cycles or Stationary Sunspot Equilibria. The present method of analysis can easily be transposed, along lines that deserve a few remarks.

First, the conceptual difficulty that has been mentioned above, and that has justified the introduction of a second version of the CK assumption, remains and suggests again reference to variants of CK2.

Clearly, the local CK restriction that serves to test SR must refer to beliefs that are compatible with and close to the equilibrium beliefs. This means that the appropriate restriction for cycles must state that the system evolves in neighborhoods that are different according to whether the date is odd or even; this means that the CK restriction on beliefs, when LSR of Sunspot Equilibria is considered, must refer to a single and well defined sunspot phenomenon.[56]

[56] This echoes the idea that Sunspot Equilibria can only be weakly-EStable, in the sense and as argued by Evans-Honkapohja (1994).

As the equations defining cycles involve a one-dimensional variable (cycles are zero of the equation $\varphi^2 = 0$, where φ^2 is the second iterate of φ), it is unsurprising that local SR of cycles involve conditions similar to those that have been obtained for Steady States (the derivative of φ^2 smaller than one, plus some homogeneity considerations; see Negroni (1998)). LSR of sunspot equilibria can be analyzed in the same way.[57]

8.3.2. *Models with Memory*

Solutions of the model with memory introduced above are trajectories that depend on the initial conditions (the steady state may only be asymptotically learnt). In the saddle path case, it is usually considered that the right solution is the trajectory converging to the Steady State, i.e. the determinate trajectory. What does the present analysis say? Let me sketch the answers.

The present case provides a vivid illustration of what has been said about the "topology" in which local restrictions are stated. A CK restriction stating that the path will remain close to the equilibrium path, in the sense that each point of the trajectory will remain close to the equilibrium point (this is "proximity" in the sense of C_0 topology), is without power. What we need are restrictions bearing on the slope of the trajectories, the rate of growth (which involves "proximity" in the C_1 topology). The CK restriction then takes the form: it is CK that the growth rate of the actual trajectory is close to the equilibrium growth rate of the determinate trajectory.[58]

With such restrictions, the kind of mental process under consideration here converges under conditions that have been exhibited in Evans-Guesnerie (1999), when all the $\alpha(i)$ have the same sign. The answer, however depends very much on whether the agents' actions at time t, which determine y_t, can be made conditional on y_t, or not. In the first case, case 1, agents' actions, conditional on y_t, can be submitted to an auctioneer, while in the other case, case 2, agents have to guess y_t, when choosing to act, i.e. make unconditional orders.

LSR OF SADDLE PATHS: *In case* 1, *the saddle-path trajectory is LSR*, *under CK restrictions on the growth rates*.[59]

In case 2, *under the same CK restrictions*, *the saddle-path trajectory is LSR only when* $\alpha\beta \geq -3/4$.

If one remembers that the saddle path solution obtains for $|\alpha+\beta| \leq 1$, then one can visualize the stability zones in the space α, β. This is done on Figure 5, where in addition, other solutions and/or bifurcations that would arise in a nonlinear version of the model, when local coefficients pass certain boundaries, are indicated.

57 Existing analyses bear upon IE-Stability of a somewhat special dynamical system (Evans-Honkapohja (1994)) or of OLG type systems (Negroni (1998)).

58 This is an exact restriction if used in the linear model or an asymptotic restriction in the nonlinear model.

59 Hence, under a neighborhood restriction with C^1 like topology. See also Gauthier (2001).

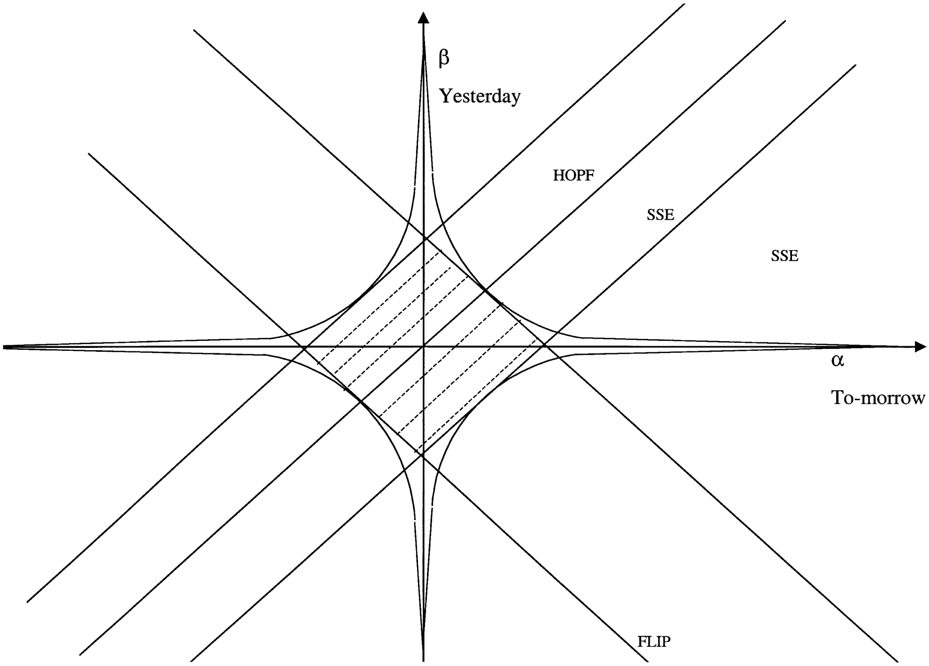


FIGURE 5

8.3.3. *Intuitions*

Let us attempt to stress lessons with intuitive appeal from the present results, although at this stage, it is a slightly risky exercise. Three "general intuitions" are proposed here. The first two seem reasonably secure, while the third one is less assured.

GI7 extrapolates from the statement of Subsection 8.2.

GI7: *Coordination is more likely when agents' actions today do not react too much to expectations about the future and when reactions to expectations are not too heterogenous.*

GI8 emphasizes that the coordination requirements (particularly in view of the demanding CK restrictions) are in some sense much stronger in intertemporal models.

GI8: *Coordination in models with long horizon requires that initial restrictions of beliefs bear on a rather long horizon.*

When the stationary equilibrium under consideration is complex, its nature must be well understood and fit a priori beliefs.

GI9 extrapolates on our last statement.

GI9: *When expectations bear on growth rates and concern saddle-path-like trajectories, coordination may be affected by coordination problems internal to the "present" period, in which cases agents' actions must not react too much either to the past or to the future*.

9. CONCLUSION

This paper has proposed a unifying viewpoint for the problem of expectational coordination. The analysis reached various degrees of generality depending on the models under consideration. Hopefully, the attempt has convinced the reader that the methodology has a broad (universal?) scope[60] and is conceptually sound. It calls for three final remarks.

First, the viewpoint offers a systematic approach to the problem of expectational coordination. Its connections with existing literature of different inspiration have already been stressed: it often enhances the rationale for studies of Iterative Expectational Stability, a concept that was proposed some time ago;[61] its results echo those of studies of coordination that have an "evolutive" perspective and complement those obtained in different fields concerned with learning.

Second, the teachings of this research strand, as well as those of competing approaches, have ultimately to be faced with facts, and by facts I mean not only artificial facts generated in learning experiments,[62] but also by empirical economic facts, that should confirm or disconfirm the sources of expectational stability derived from the theory. Discussing the issue is clearly outside the scope of this paper. But I would like to stress that the attempt to translate, sometimes boldly, formal statements into economically appealing intuitions, whatever its shortcomings, looks like a key prerequisite in the research program just evoked.

Third, and finally, and somewhat independently of the empirical evidence, the test of Local Strong Rationality (or Local Dominance Solvability) "anchoring expectations in CK" is, in my opinion, an unavoidable requirement for any theory that claims reliance on the "rationality hypothesis": it was proposed here that we call Strongly Rational a Rational Expectations Equilibrium that meets the CK stability requirement; a more extreme view is that an equilibrium that does not pass some version of the local test may be wrongly called "rational"! This is, I

[60] We do not claim to have covered all existing applications.

For example, in the context of an intertemporal general equilibrium, Guesnerie-Hens (1998), following Ghosal (1994), compare the conditions of SR with the conditions guaranteeing the convergence of an algorithm of computation of Arrow-Debreu prices in an intertemporal setting, which uses intra-period market clearing and has been proposed by Balasko (1994). See also Ghosal (2001).

Other work that relates with that presented here includes Cho (1992) and Frankel-Pauzner (2000).

[61] Although it has been more or less given up, existing studies have rather concentrated on a competing and different concept of Differential E-Stability.

[62] A number of possible experiments are clearly suggested by the above analysis.

understand a debatable opinion, but, as such, may be suitable as a speculative conclusion.

DELTA (*Joint Research Unit CNRS*, *EHESS*, *ENS*), 48 *Boulevard Jourdan*, *ENS*, 75014, *Paris*, *France*; *guesnerie@delta.ens.fr*

Manuscript received October, 1999; final revision received June, 2001.

APPENDIX: PROOF OF STATEMENT IN PART II, SECTION 5

The sequential equilibrium associated with the simultaneous equilibrium $\overline{X}$ consists of $\overline{X}/2$, $\overline{X}/2$; w.l.o.g one can take $\overline{X}=0$.

Let us now describe, in line with Axiom 1, how the sequential system reacts to (small) changes in expectations: if x_1 is the first period outcome, $x_1 \neq 0$ but close to zero, and if mean beliefs of agent i on the others' actions in period 2 are $x_2(i)$, then the second period outcome will approximately be

$$x_2' = \int_{I_2} A(i)(x_1 + x_2(i))\, di$$

where I_2 (resp. I_1) is the set of agents deciding at period 2 (resp. 1).

In particular, the period 2 equilibrium obtains as

$$x_2 = \int_{I_2} A(i)(x_1 + x_2)\, di,$$

i.e.

$$x_2 = (I - A/2)^{-1}(A/2)x_1.$$

Let us now evaluate the LSR of the second period equilibrium.

Substracting the previous equalities, we have $x_2' - x_2 = \int_{I_2} A(i)(x_2(i) - x_2)\, di$. If $x_2(i) - x_2 \in b$, a ball whose radius, evaluated with the Euclidean norm on $\mathcal{B}$, the basis of A, is 1, then it is CK that $x_2' - x_2 \in (\int_{I_2} n(i)\, di)b = 1/2(\int_I n(i)\, di)\dots$ etc. Hence we have conclusion (i).

Now let us assume that, at the first period, it is CK that $x_1 \in b$.

If agent i has the average belief $x_1(i)$ for x_1, then he believes, on average, that $x_2 = x_2(i) = (I - A/2)^{-1}(A/2)x_1(i)$, and then that $x_1 + x_2 = (I + (I - A/2)^{-1}(A/2))x_1(i)$. It follows, since the matrix of the right-hand side has the same basis as A, that the aggregate outcome x' belongs to $(\int_{I_1} n(i)\, di)(I + (I - A/2)^{-1}(A/2))b$.

Now, if λ is an eigenvalue of A, $1/(1-\lambda/2)$ is an eigenvalue of $(I + (I - A/2)^{-1}(A/2))$. Calling S_A the spectrum of A, $\rho(\lambda)$ the modulus of λ, we have

$$x' \in \left[\int_{I_1} n(i)\, di)\right]\left[\max_{\lambda \in S_A} \rho(1/(1-\lambda/2))\right].$$

We note that $1/(1+\rho(x)) \leq \max_x \rho(1/1-x) \leq 1/(1-\rho(x))$.

Now if, $\int_I n(i)\, di = 2\int_{I_1} n(i)\, di \leq 1$, then, we know from previous discussion, that

$$\max_{\lambda \in S_A} \rho(\lambda) \leq 1 \quad \text{and} \quad \left[\int_{I_1} n(i)\, di\right]\left[\max_{\lambda \in S_A} \rho(1/1-\lambda/2)\right] \leq 1/2\left(\int_I n(i)\, di\right).$$

Conclusion (ii) follows.

Finally, if A has eigenvalues with a negative real part,

$$\max_{\lambda \in S_A} \rho(1/1-\lambda) \leq 1, \quad \text{and} \quad \int_{I_2} n(i)\, di \max_{\lambda \in S_A} \rho(1/1-\lambda/2) \leq 1/2\left(\int_I n(i)\, di\right). \qquad \textit{Q.E.D.}$$

REFERENCES

AUMANN, R., AND A. BRANDENBURGER (1995): “Epistemic Conditions for Nash Equilibrium,” *Econometrica*, 63, 1161–1180.

AZARIADIS, C. (1993): *Intertemporal Macroeconomics*. Cambridge, Mass. and Oxford: Blackwell.

BALASKO, Y. (1994): “The Expectational Stability of Walrasian Equilibria,” *Journal of Mathematical Economics*, 23, 179–203.

BASU, K., AND J. WEIBULL (1991): “Strategy Subsets Closed under Rational Behaviour,” *Economic Letters*, 36, 141–146.

BERNHEIM, B. D. (1984): “Rationalizable Strategic Behavior,” *Econometrica*, 52, 1007–1028.

BORGERS, T. (1992): “Iterated Elimination of Dominated Strategies in a Bertrand Edgeworth Model,” *Review of Economic Studies*, 59, 163–176.

BOSSAERTS, P. (1997): “Rational Expectations Equilibrium when Priors are Inconsistent,” Mimeo, Caltech.

BRAY, M. (1982): “Learning, Estimation and the Stability of Rational Expectations Equilibria,” *Journal of Economic Theory*, 26, 318–339.

BRAY, M., AND D. KREPS (1988): “Rational Learning and Rational Expectations,” in *Arrow and the Ascent of Economic Theory*, ed. by G. Feiwell. London: Macmillan, Chap. 19, pp. 597–625.

BRYANT, J. (1983): “A Simple Rational Expectations Keynes-Type Model,” *Quarterly Journal of Economics*, 98, 525–529.

——— (1984): “An Example of a Dominance Approach to Rational Expectations,” *Economics Letters*, 16, 249–255.

——— (1987): “The Paradox of Thrift, Liquidity Preferences and Animal Spirits,” *Econometrica*, 55, 1231–1237.

CALVO, G. A. (1983): “Comments on Rational Expectations in Macroeconomic Models,” in *Individual Forecasting and Aggregate Outcomes*, ed. by R. Frydman and E. Phelps. Cambridge: Cambridge University Press.

CARLSSON, H., AND E. VAN DAMME (1993): “Global Games and Equilibrium Selection,” *Econometrica*, 61, 998–1018.

CASS, D., AND K. SHELL (1983): “Do Sunspots Matter?” *Journal of Political Economy*, 91, 409–426.

CHAMLEY, C., AND D. GALE (1994): “Information Revelation and Strategic Delay in a Model of Investment,” *Econometrica*, 62, 1065–1085.

CHIAPPORI, P. A., P. Y. GEOFFARD, AND R. GUESNERIE (1992): “Sunspots Fluctuations around a Steady State: The Case of Multidimensional, One-Step Forward Looking Models,” *Econometrica*, 60, 1097–1126.

CHIAPPORI, P. A., AND R. GUESNERIE (1989): “On Stationary Sunspot Equilibria of Order k,” in *Economic Complexity: Chaos Sunspots, Bubbles and Nonlinearity*, Proceedings of the Austin Conference on Economic Complexity, ed. by W. Barnett, J. Geweke, and K. Shell. Cambridge: Cambridge University Press, pp. 21–44.

——— (1991): “Sunspot Equilibria in Sequential Markets Models,” in *Handbook in Mathematical Economics*, 4, ed. by W. Hildenbrand and H. Sonnenschein. Amsterdam: North Holland, pp. 1683–1762.

CHO, IN KOO (1992): “Stationarity, Rationalizability and Bargaining,” Mimeo, University of Chicago.

COOPER, R., AND A. JOHN (1988): “Coordinating Coordination Failures in Keynesian Macroeconomics,” *Quarterly Journal of Economics*, 103, 441–463.

DAVILA, J. (1997): “Sunspot Equilibria in Dynamics with Predetermined Variables,” *Economic Theory*, 10, 483–495.

DE CANIO, S. (1979): “Rational Expectations and Learning from Experience,” *Quarterly Journal of Economics*, 93, 47–58.

DE CASTRO, S. (1997): “A Rationalizable Expectations Criterion for Monetary Policy in the Macroeconomic Game of Conflict over the Distribution,” Mimeo, IHD, Brasilia.

DESGRANGES, G. (1999): “Révélation d’Information par les Prix et Connaissance Commune de la Rationalité,” Thesis, EHESS.

DESGRANGES, G., P. Y. GEOFFARD, AND R. GUESNERIE (2000): "The Transmission of Information through Prices: An Alternative Exploratory Assessment of Market Performance," Mimeo, Delta, 2000-27.

DESGRANGES, G., AND R. GUESNERIE (2000): "Common Knowledge and the Information Revealed through Prices: Some Conjectures," in *Price Expectations in Goods and Financial Markets*, ed. by F. Gardes and G. Prat. Paris: Edward Elgar, pp. 52–62.

EVANS, G. (1985): "Expectational Stability and the Multiple Equilibrium Problem in Linear Rational Expectations Models," *Quarterly Journal of Economics*, 100, 1217–1233.

EVANS, G., AND R. GUESNERIE (1993): "Rationalizability, Strong Rationality and Expectational Stability," *Games and Economic Behavior*, 5, 632–646.

——— (1999): "Coordination on Saddle Path Solutions: The 'Eductive' Viewpoint 1-Linear Univariate Models," Delta DP 9915.

EVANS, G., AND S. HONKAPOHJA (1994): "On the Local Stability of Sunspot Equilibria under Adaptive Learning Rules," *Journal of Economic Theory*, 64, 142–161.

——— (1998): "Economic Dynamics with Learning: New Stability Results," *Review of Economic Studies*, 65, 23–44.

——— (1999): "Learning Dynamics," in the *Handbook in Macroeconomics*, ed. by J. Taylor and M. Woodford. Princeton, N.J.: Princeton University Press, Ch. 7.

FARQHARSON, R. (1969): *Theory of Voting*. New Haven: Yale University Press.

FRANKEL, D., AND A. PAUZNER (2000): "Resolving Indeterminacy in Dynamic Settings: The Role of Shocks," *Quarterly Journal of Economics*, 115, 285–304.

FOURGEAUD, C., C. GOURIEROUX, AND J. PRADEL (1986): "Learning Procedures and Convergence to Rationality," *Econometrica*, 54, 845–868.

FUDENBERG, D., AND D. LEVINE (1998): *The Theory of Learning in Games.* Cambridge, MA: MIT Press.

GABAY, D., AND H. MOULIN (1980): "On the Uniqueness and Stability of Nash Equilibria in Non Cooperative Games," in *Applied Stochastic Control in Econometrics and Management Science*, ed. by A. Bensoussan, P. Kleindorfer, and N. Tatierd. Amsterdam: North Holland.

GAUTHIER, S. (2001): "On the Dynamic Equivalence Principle in Linear Rational Expectations Models," forthcoming in *Macroeconomic Dynamics*.

GEANAKOPLOS, J. (1992): "Common Knowledge," *Journal of Economic Perspectives*, 6, 53–82.

GHOSAL, S. (1994): "Eductive Stability and T-Stability in an Intertemporal Model," Mimeo, University of Warwick.

——— (2001): "Sequential Coordination, Eductive Stability and Rationalizability," Mimeo, University of Warwick.

GRANDMONT, J. M., AND G. LAROQUE (1986): "Stability of Cycles and Expectations," *Journal of Economic Theory*, 40, 138–151.

GROSSMAN, S. (1976): "On the Efficiency of Competitive Stock Markets where Traders have Diverse Information," *Journal of Finance*, 31, 573–585.

GUESNERIE, R. (1986): "Stationary Sunspot Equilibria in an *n* Commodity World," *Journal of Economic Theory*, 40, 103–104.

——— (1989): "A Propos de la Rationalité des Anticipations Rationnelles," in *Théorie Economique et Crise des Marchès Financiers*, ed. By P. Artus and F. Bourguinat. Paris: Economica, 31–43.

——— (1992): "An Exploration of the Eductive Justifications of the Rational Expectations Hypothesis," *American Economic Review*, 82, 1254–1278.

——— (1993): "Theoretical Tests of the Rational Hypothesis in Economic Dynamical Models," *Journal of Economic Dynamics and Control*, 17, 847–864.

——— (2001): "Short Run Expectational Coordination: Fixed versus Flexible Wages," *Quarterly Journal of Economics*, 116, 1115–1147.

GUESNERIE, R., AND T. HENS (1998): "Expectational Stability in a Two-Period Exchange Economy," Mimeo, Delta.

GUESNERIE, R., AND J.-C. ROCHET (1993): "(De) Stabilizing Speculation: An Alternative Viewpoint," *European Economic Review*, 37, 1043–1063.

GUESNERIE, R., AND M. WOODFORD (1991): "Stability of Cycles with Adaptive Learning Rules," in *Equilibrium Theory and Applications: Proceedings of the Sixth International Symposium in Economic Theory and Econometrics*, ed. by W. A. Barnett et al. Cambridge: Cambridge University Press, pp. 110–133.

GUESNERIE, R., AND M. WOODFORD (1992): "Endogenous Fluctuations," in *Advances in Economic Theory*, Econometric Society Monograph, ed. by J. J. Laffont. Cambridge: Cambridge University Press, pp. 289–412.

HEINEMANN, F. (1997): "Ratonalizable Expectations and Sunspot Equilibria in an Overlapping Generations Economy," *Journal of Economics*, 65, 257–277.

HEINEMANN, M. (2001): "Existence of Strongly Rational Expectations Equilibria on Asset Markets with Asymmetric Information," Mimeo, University of Hannover.

HORN, A., AND C. JOHNSON (1985): *Matrix Analysis*. Cambridge: Cambridge University Press.

JANSSEN, M. (1991): "A Multi Market Bertrand Game as a Foundation of Walrasian Equilibrium," Mimeo, University of Groningen.

KAJII, A., AND S. MORRIS (1997): "The Robustness of Equilibria to Incomplete Information," *Econometrica*, 60, 1283–1309.

LUCE, R., AND H. RAIFFA (1957): *Games and Decisions*. New York: Wiley.

MARCET, A., AND J. SARGENT (1989): "Convergence of Least Squares Learning Mechanisms in Self Referential Linear Stochastic Models," *Journal of Economic Theory*, 48, 337–368.

MARIMON, R. (1996): "Learning from Learning in Economics," in *Advances in Economic Theory*, ed. by D. Kreps and K. Wallis. Cambridge: Cambridge University Press.

MILGROM, P., AND J. ROBERTS (1991): "Rationalizability, Learning and Equilibrium in Games with Strategic Complementarities," *Econometrica*, 58, 1255–1277.

MILGROM, P., AND C. SHANNON (1994): "Monotone Comparative Statics," *Econometrica*, 62, 157–180.

MORRIS, S., R. ROB, AND H. S. SHIN (1995): "p-Dominance and Belief Potential," *Econometrica*, 63, 145–157.

MOULIN, H. (1979): "Dominance Solvable Voting Schemes," *Econometrica*, 47, 1337–1351.

——— (1981): *Théorie des Jeux pour l'Economie et la Politique*. Paris: Hermann.

MUTH, J. (1961): "Rational Expectations and the Theory of Price Movements," *Econometrica*, 29, 315–335.

NAGEL, R. M. (1995): "Unraveling in Guessing Game: An Experimental Study," *American Economic Review*, 85, 1313–1326.

NEGRONI, G. (1998): "Strong Rationality of Sunspot Equilibria and Cycles," Mimeo, Delta.

PEARCE, D. (1984): "Rationalizable Strategic Behavior and the Problem of Perfection," *Econometrica*, 52, 1029–1050.

POINCARÉ, H. (1901): Lettre à L. Walras, published in "Bulletin de la Societé Vaudoise de Sciences Naturelles," 45 (1909).

RADNER, R. (1979): "Rational Expectations Equilibria: Generic Existence and the Information Revealed by Prices," *Econometrica*, 47, 655–678.

TAN, T., AND S. R. COSTA DA WERLANG (1988): "On Aumann's Notion of Common Knowledge, an Alternative Approach," *Journal of Economic Theory*, 45, 370–391.

TOPKIS, D. (1979): "Equilibrium Points in Nonzero-Sum *n*-Person Submodular Games," *Siam Journal of Control and Optimization*, 17, 773–787.

TOWNSEND, R. M. (1978): "Market Anticipations, Rational Expectations and Bayesian Analysis," *International Economic Review*, 19, 481–494.

VIVES, X. (1990): "Nash Equilibria with Strategic Complementarities," *Journal of Mathematical Economics*, 19, 305–321.

——— (1993): "How Fast do Rational Agents Learn?" *Review of Economic Studies*, 60, 329–347.

WOODFORD, M. (1990): "Learning to Believe in Sunspots," *Econometrica*, 58, 277–307.

YOUNG, H. PEYTON (1996): "The Economics of Conventions," *Journal of Economic Perspectives*, 10, 105–122.