Macroeconomic Dynamics, **7**, 2003, 171–191. Printed in the United States of America.
DOI: 10.1017.S1365100501010264

ARTICLES

UNIQUENESS OF BUBBLE-FREE SOLUTION IN LINEAR RATIONAL EXPECTATIONS MODELS

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One usually identifies bubble solutions to linear rational expectations models by extra components (irrelevant lags) arising in addition to market fundamentals. Although there are still many solutions relying on a minimal set of state variables, i.e., relating in equilibrium the current state of the economic system to as many lags as initial conditions, there is a conventional wisdom that the bubble-free (fundamentals) solution should be unique. This paper examines the existence of endogenous stochastic sunspot fluctuations close to solutions relying on a minimal set of state variables, which provides a natural test for identifying bubble and bubble-free solutions. It turns out that only one solution is locally immune to sunspots, independently of the stability properties of the perfect-foresight dynamics. In the standard saddle-point configuration for these dynamics, this solution corresponds to the so-called saddle stable path.

Keywords: Rational Expectations, Bubbles, Sunspots, Saddle-Path Property

1. INTRODUCTION

It is now well known that the rational expectations hypothesis generally does not pick out a unique equilibrium path. Consequently, one usually introduces into analysis additional selection devices that give an account of the relevance of special paths. The aim of such criteria is often to rule out bubble solutions, that is, paths that are determined in particular by traders' expectations. Although there are cases in which the identification of bubble solutions and bubble-free (fundamentals) solutions is unquestionable, a sampling of the literature [Flood and Garber (1980), Burmeister et al. (1983), and, more recently, McCallum (1999), among others] suggests that there is still no agreement on what should be a bubble. Thus, the purpose of the present paper is to progress toward defining bubble and bubble-free

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 1365-1005/03 $12.00

solutions in linear economies in which agents forecast only one period ahead, and in which the number of predetermined variables is arbitrary, but fixed.

In the case of explicit justifications, the most often used criterion is that of stability or nonexplosiveness of endogenous variables [Blanchard and Kahn (1980), Blanchard and Fischer (1987), and Sargent (1987)]. From a practical point of view, attention typically is restricted to a particular configuration in which this criterion provides a unique outcome, namely, the so-called saddle-point configuration for the perfect-foresight dynamics, characterized by a number of stable roots that is equal to the number of initial conditions (the number of predetermined variables). More precisely, the model features a continuum of paths where the endogenous variable explodes toward infinity and only one saddle stable path where it remains bounded and even converges toward the stationary state. The dynamics restricted to the saddle stable path make the current state dependent upon a number of lags that is equal to the number of predetermined variables. Because of this property, it is usually asserted that market fundamentals entirely determine the actual path of the economy and expectations do not play a role in this equilibrium. However, the stability criterion fails to select a single solution as soon as there are more stable roots than predetermined variables, that is, in the so-called indeterminate configuration for the perfect-foresight dynamics.

The minimal-state variable (MSV) criterion from McCallum (1983) is conceived to apply also in this case. It recommends the elimination of solutions in which the current state relies on a number of lags larger than the number of predetermined variables, that is, solutions that display an extra component arising in addition to the components that reflect market fundamentals. Such solutions are said to be with a bubble, given that traders' expectations necessarily matter. There is a large agreement on ruling out these solutions and focusing attention on the solutions with a minimum number of lags, that is, the solutions such that the number of lags is equal to the number of predetermined variables. However, in general models, there still remain many solutions involving a minimal number of lags, whereas there is a conventional wisdom that there should be a unique solution, termed bubble-free. Hence, an additional device is needed to identify only one solution. McCallum (1999) proposes then to introduce a subsidiary principle that is, at first sight, unrelated to the definition of a bubble in general models [see d'Autume (1990) for a discussion]. As McCallum (1999) emphasizes, such an augmented MSV criterion "identifies a single solution that can *reasonably* [emphasis added] be interpreted as the unique solution that is free of bubble components, i.e., the fundamentals solution." Precisely, this MSV criterion requires that the equilibrium path involve a minimal number of lags, whatever the values of the exogenous parameters are, that is, even in degenerate cases in which some of them are equal to zero. In linear models, it appears that this condition always selects a unique solution. In particular, in the saddle-point configuration, the MSV solution is the equilibrium path that corresponds to the saddle stable path. In what follows, we call "McCallum's conjecture" the claim that the MSV solution is the (unique) solution deserving to be called bubble-free (or the fundamentals solution).

To discuss this conjecture, we study the existence of self-fulfilling sunspot-like beliefs, the support for which is very close to solutions with a minimal number of lags. Namely, we consider that the existence of such sunspot equilibria accounts for expectations that matter close to these solutions, and therefore, to deserve to be bubble-free, a solution should be free of any neighboring sunspot equilibria. Our results are then in accordance with McCallum's conjecture: Sunspot fluctuations never arise close to the MSV solution, and they may occur arbitrarily close to any other solution that displays a minimal number of lags. These results are shown to hold in general (univariate) linear models where agents forecast only one period ahead, and with an arbitrary number $L \geq 0$ of predetermined variables. In this framework, the dynamics with perfect foresight are locally governed by $(L+1)$ perfect-foresight (growth rates) roots $\lambda_1, \ldots, \lambda_{L-1}$ with $|\lambda_1| < \cdots < |\lambda_{L+1}|$. Hence, there are $(L+1)$ solutions where the current state depends on only L lags. Each one corresponds to an equilibrium path that belongs to the eigensubspace spanned by L eigenvectors associated with L among $(L+1)$ perfect-foresight roots; that is, all of these paths are defined by only L coefficients. In particular, in the saddle-point configuration $(|\lambda_L| < 1 < |\lambda_{L+1}|)$, the saddle stable path is governed by the L roots of lowest modulus $\lambda_1, \ldots, \lambda_L$. In this configuration as well as in any other, McCallum's (1999) conjecture is that this latter solution is actually the unique bubble-free solution; that is, beliefs are not relevant in this solution but they should generically matter for paths corresponding to L other roots (and including, in particular, the root of largest modulus λ_{L+1}). To discuss this assertion, we assume that agents observe an exogenous sunspot process that does not affect fundamentals, and that they hold beliefs that are correlated to the sunspot process and consist in randomizing over paths arbitrarily close to solutions with L lags, that is, over paths defined by L coefficients arbitrarily close to the L coefficients that define solutions with a minimal number of lags. We show that (i) beliefs can never be self-fulfilling in the neighborhood of the path that is governed by the L roots of lowest modulus $\lambda_1, \ldots, \lambda_L$ and (ii) for any other solution with a minimal number of lags, there always exist sunspot processes ensuring that some beliefs are self-fulfilling.

This paper is organized as follows: In Section 2, we present our results in the simple benchmark framework also considered by McCallum (1999), where $L = 1$. Then, in Section 3, we tackle the general case in which $L \geq 0$ is arbitrary. A brief summary of the results is given in Section 4.

2. PRELIMINARY EXAMPLE

The reduced form that we first consider supposes that the current equilibrium state is a scalar x_t linked with both the common forecast of the next state $E(x_{t+1} \mid I_t)$ (where E denotes the mean operator and I_t the information set of agents at date t) and the predetermined state x_{t-1} through the following temporary equilibrium map:

$$\gamma E(x_{t+1} \mid I_t) + x_t + \delta x_{t-1} = 0, \qquad (1)$$

where the real numbers γ and δ represent the relative weights of future and past, respectively. Equation (1) stands for a first-order approximation of temporary equilibrium dynamics in a suitable neighborhood $V(\bar{x})$ of a locally unique stationary state $\bar{x}$ whose value is normalized to zero. This formulation is general enough to encompass equilibrium conditions of simple versions of overlapping generations economies with production [Reichlin (1986)], and those of infinite-horizon models with cash-in-advance constraints [Woodford (1986), Bosi and Magris (1997)]. It is also commonly used as a benchmark case in the temporary equilibrium literature [Grandmont and Laroque (1990, 1991), Grandmont (1998)]. It serves the purpose of McCallum (1999). In this model, the local perfect-foresight dynamics rely on two local perfect-foresight roots, λ_1 and λ_2 (with $|\lambda_1| < |\lambda_2|$ by definition); that is, there are two paths along which the current state x_t is determined by only one lag x_{t-1} through a constant growth rate (factor) x_t/x_{t-1} equal to either λ_1 or λ_2. In such paths, traders' forecasts do not *a priori* matter because the number of lags that affect the current state is equal to the number of predetermined variables. The path corresponding to λ_1 (the λ_1 path, for convenience) governs the perfect foresight restricted to the saddle stable branch in the saddle-point case ($|\lambda_1| < 1 < |\lambda_2|$). The issue is whether this λ_1 path is indeed the only one that is bubble-free, as claimed by McCallum (1999). To tackle this problem, we build a sunspot process over growth rates arbitrarily close to the perfect-foresight roots λ_1 and λ_2. The existence of the sunspot equilibria so defined provides a clear method for defining bubbles. Actually, it turns out that such expectations-driven fluctuations do not arise close to the λ_1 path but that they do occur close to the λ_2 path, independently of the stability (determinacy) properties of the local perfect-foresight dynamics. As a result, the λ_1 path is the single solution of model (1) that can be termed bubble-free.

2.1. Deterministic Rational Expectations Equilibria

A local perfect-foresight equilibrium is a sequence of state variables $\{x_t\}_{t=-1}^{\infty}$ associated with the initial condition x_{-1}, and such that the recursive equation (1) with $E(x_{t+1} \mid I_t) = x_{t+1}$ holds at all times:

$$\gamma x_{t+1} + x_t + \delta x_{t-1} = 0. \tag{2}$$

Consequently, the current state may be related to either one or two lags in (2). In the latter case, the solution is $x_t = -(1/\gamma)x_{t-1} - (\delta/\gamma)x_{t-2}$. It displays more lags than predetermined variables. It is, accordingly, a bubble solution. On the contrary, the state variable is obedient in the former case to the law of motion $x_t = \beta x_{t-1}$, where β satisfies $\gamma\beta^2 x_{t-1} + \beta x_{t-1} + \delta x_{t-1} = 0$ for any $x_{t-1} \in V(\bar{x})$; that is, β is a root λ_i ($i = 1, 2$) of the characteristic polynomial associated with (2). Throughout the paper, we assume that λ_1 and λ_2 (with $|\lambda_1| < |\lambda_2|$) are real. For these two solutions, $x_t = \lambda_i x_{t-1}$ ($i = 1, 2$), the number of lags is equal to the number of predetermined variables, and the fundamentals (γ, δ) and the initial condition x_{-1} are then sufficient to determine the actual path of the economy; that is, forecasts play *a priori* no role. Neither of these two paths has an *a priori* special characteristic that would justify labeling it as bubble-free. Nevertheless the

λ_1 path is usually presumed to be the unique solution where bubbles are absent. In particular, this claim holds true according to the MSV criterion of McCallum (1999). Namely, in the case $\delta = 0$—that is, if no predetermined variables enter the model—λ_1 reduces to 0 (and the λ_1 path reduces to the steady state $x_t = \bar{x}$), whereas λ_2 does not. This implies that the current state is not linked to past realizations along the λ_1 path, but it is along the λ_2 path. The λ_1 path is therefore the only solution displaying a minimal number of lags whatever the values γ of δ are; this is precisely the definition of the MSV solution.

2.2. Stochastic Sunspot Rational Expectations Equilibria

The purpose of this section is to show that traders' beliefs do not matter (do matter) in the immediate vicinity of the λ_1 path (λ_2 path) when $\delta \neq 0$, which provides a simple basis for the choice of bubble-free trajectories. We assume that agents observe a public exogenous sunspot signal with two different states, $s_t = 1, 2$ at every date $t \geq 0$. The signal follows a discrete-time Markov process with stationary transition probabilities. Let Π be the two-dimensional transition matrix whose ss'th entry $\pi_{ss'}$ is the probability of sunspot signal s' at date $t+1$ when the signal is s at date t. Agents believe that rates of growth are perfectly correlated with the exogenous stochastic process. Let $\beta_s (s = 1, 2)$ be the guess of the rate of growth whenever signal s is observed at the outset of a given period; that is, agents deduce from the occurrence of signal s at date t that x_t should be determined according to the following law of motion:

$$x_t = \beta_s x_{t-1}. \tag{3}$$

At date t, the information set includes all past realizations of the state variable and of the sunspot signal; that is, $I_t = \{x_{t-1}, \ldots, x_{-1}, s_t, \ldots, s_0\}$. Although I_t does not contain x_t, we will consider that agents' expectations at date t are made conditionally to x_t; that is, agents believe that x_{t+1} will be equal to $\beta_{s'} x_t$ with probability $\pi_{ss'}$. This way of forming expectations is made for technical simplicity. It influences none of our results, which bear on stationary equilibrium only (as defined later); that is, at equilibrium, beliefs are self-fulfilling and the actual x_t is always equal to its expected value at date t (i.e., $\beta_s x_{t-1}$). As a result, the expected value $E(x_{t+1} \mid I_t)$ is written

$$x_{t+1}^e = E(x_{t+1} \mid s_t = s) = \left[\sum_{s'=1}^{2} \pi_{ss'} \beta_{s'}\right] x_t \equiv \bar{\beta}_s x_t, \tag{4}$$

where $\bar{\beta}_s$ represents the (expected) average growth rate between t and $(t+1)$ conditionally to the event $s_t = s$. The actual dynamics are obtained by reintroducing expectations (4) into the temporary equilibrium map (1). If s occurs at date t, then the actual law of motion of the state variable satisfies

$$\begin{aligned} &\gamma \bar{\beta}_s x_t + x_t + \delta x_{t-1} = 0 \\ &\quad \Leftrightarrow x_t = -[\delta/(1 + \gamma \bar{\beta}_s)]\, x_{t-1} \equiv \Omega_s(\beta_1, \beta_2) x_{t-1}. \end{aligned} \tag{5}$$

We are now in a position to define a two-state sunspot equilibrium on growth rate, hereafter denoted SSEG(k, L), where k is the number of different signals of the sunspot process and L represents the number of lags taken into account by agents. In this section, we thus have $k = 2$ and $L = 1$.

DEFINITION 1. *A two-state stationary sunspot equilibrium on growth rate* [*denoted an SSEG*(2, 1)] *is a pair* $(\boldsymbol{\beta}, \Pi)$ *where* $\boldsymbol{\beta}$ *is a two-dimensional vector* $(\boldsymbol{\beta_1}, \boldsymbol{\beta_2})$ *and* Π *is the two-dimensional stochastic matrix that triggers beliefs of traders, such that* (*i*) $\beta_1 \neq \beta_2$ *and* (*ii*) $\beta_s = \Omega_s(\beta_1, \beta_2)$ *for* $s = 1, 2$.

At an SSEG(2, 1), the expected growth rate β_s used in (3) is self-fulfilling whatever the current sunspot signal s is; that is, β_s coincides with the actual growth rate $\Omega_s(\beta_1, \beta_2)$ given in (5). The economy will indurate endogenous stochastic fluctuations as soon as condition (*i*) is satisfied. In the case in which this condition fails, one can speak of a *degenerate* SSEG(2, 1). Degenerate SSEG (2, 1) are pairs $((\lambda_s, \lambda_s), \Pi)$ where λ_s is a perfect foresight growth rate and the transition matrix Π is arbitrary: growth rate remains constant through time and beliefs are self-fulfilling, whatever the sunspot process is.

Formally speaking, we shall say that a neighborhood of a SSEG(2, 1), denoted $((\beta_1, \beta_2), \Pi)$, is a product set $V \times \mathcal{M}_2$, where V is a neighborhood of (β_1, β_2) for the natural product topology on $\mathbf{R}^2$ and $\mathcal{M}_2$ is the set of all the 2-dimensional stochastic matrices Π. Then, we shall say that another SSEG (2, 1), denoted $((\beta_1', \beta_2'), \Pi')$, is in the neighborhood of $((\beta_1, \beta_2), \Pi)$ [respectively, $(\lambda_s, \lambda_{s'})$] whenever the vector $(\boldsymbol{\beta_1'}, \boldsymbol{\beta_2'})$ stands close to (β_1, β_2) [$(\lambda_s, \lambda_{s'})$] and whatever the matrices Π and Π' are. The next result is from a study of the existence of SSEG(2, 1) in the neighborhood of a λ_s path ($s = 1, 2$), that is, such that (β_1, β_2) stands close enough to (λ_s, λ_s).

PROPOSITION 1. *Let* $\gamma \neq 0$ *and* $\delta \neq 0$. *Then there is a neighborhood of* (λ_1, λ_1) *in which no SSEG*(2, 1) *exist but SSEG*(2, 1) *do exist in every neighborhood of* (λ_2, λ_2).

Proof. Let us define the map Ω from $\mathbf{R}^2$ onto $\mathbf{R}^2$ in the following way:

$$(\beta_1, \beta_2) \to \Omega(\beta_1, \beta_2) = (\Omega_1(\beta_1, \beta_2) - \beta_1, \Omega_2(\beta_1, \beta_2) - \beta_2),$$

so that an SSEG(2, 1) is characterized by $\Omega(\beta_1, \beta_2) = (0, 0)$ and $\beta_1 \neq \beta_2$. Let $D\Omega(\beta_1, \beta_2)$ be the two-dimensional Jacobian matrix of the map Ω calculated at point (β_1, β_2). Because λ_1 and λ_2 are the roots of the characteristic polynomial corresponding to (2), $\gamma/\delta = 1/\lambda_1\lambda_2$. This identity and some computations lead to

$$D\Omega(\lambda_s, \lambda_s) = \frac{\lambda_s^2}{\lambda_1\lambda_2}\Pi - I_2 \quad \text{for} \quad s = 1, 2,$$

with I_2 the two-dimensional identity matrix.

Notice that $\Omega(\lambda_s, \lambda_s) = (0, 0)$ for every Π. Recall that the two eigenvalues of Π are $(\pi_{11} + \pi_{22} - 1)$ and 1 [see, e.g., Chung (1967)]. The eigenvalues of $D\Omega(\lambda_s, \lambda_s)$ are therefore

$$\mu_1 = \frac{\lambda_s^2}{\lambda_1\lambda_2}(\pi_{11} + \pi_{22} - 1) - 1.$$

$$\mu_2 = \frac{\lambda_s^2}{\lambda_1\lambda_2} - 1.$$

The determinant of the Jacobian is $\det D\Omega(\lambda_s, \lambda_s) \equiv \mu_1\mu_2$. In the generic case, $\lambda_1 \neq \lambda_2$, one has $\mu_2 \neq 0$, and therefore

$$\det D\Omega(\lambda_s, \lambda_s) = 0 \Leftrightarrow \pi_{11} + \pi_{22} - 1 = \frac{\lambda_1\lambda_2}{\lambda_s^2}.$$

This last condition reduces to $\pi_{11} + \pi_{22} - 1 = \lambda_2/\lambda_1$ if $\lambda_s = \lambda_1$, and $\pi_{11} + \pi_{22} - 1 = \lambda_1/\lambda_2$ if $\lambda_s = \lambda_2$. Noticing that $|\pi_{11} + \pi_{22} - 1| < 1$ shows that $\det D\Omega(\lambda_s, \lambda_s) = 0$ is obtained for some matrices Π if and only if $s = 2$.

For the case $\lambda_s = \lambda_1$, the proposition results then from applying the implicit functions theorem to each point $((\lambda_1, \lambda_1), \Pi)$. The precise argument requires the compacity of the set of stochastic matrices Π (because a matrix Π is characterized by π_{11} and π_{22}, this set can be identified for instance to $[0, 1]^2$). It is as follows: for every matrix Π_0, there are open neighborhoods U_{Π_0} of (λ_1, λ_1) and V_{Π_0} of Π_0 and a smooth function T_{Π_0} from V_{Π_0} onto U_{Π_0} such that

$$\forall(\beta_1, \beta_2) \in U_{\Pi_0}, \quad \forall\,\Pi \in V_{\Pi_0}, \Omega_\Pi(\beta_1, \beta_2) = 0 \Leftrightarrow (\beta_1, \beta_2) = T_{\Pi_0}(\Pi). \qquad \textbf{(6)}$$

By compacity of the set of stochastic matrices Π, there is a *finite* set C of Π_0 such that $\cup_{\Pi_0 \in C} V_{\Pi_0}$ is the whole set of stochastic matrices. Hence, the family of functions T_{Π_0} for $\Pi_0 \in C$, uniquely defines a smooth function $(\beta_1, \beta_2) = T(\Pi)$ on the whole set of stochastic matrices onto the intersection $\cap_{\Pi_0 \in C} U_{\Pi_0}$. One has

$$\forall(\beta_1, \beta_2) \in \cap_{\Pi_0 \in C} U_{\Pi_0}, \quad \forall\,\Pi, \Omega_\Pi(\beta_1, \beta_2) = 0 \Leftrightarrow (\beta_1, \beta_2) = T(\Pi).$$

Given that $\Omega_\Pi(\lambda_1, \lambda_1) = 0$ holds for every Π, $T(\Pi)$ is simply equal to (λ_1, λ_1) for every Π, and there is no other (β_1, β_2) in $\cap_{\Pi_0 \in C} U_{\Pi_0}$ satisfying $\Omega_\Pi(\beta_1, \beta_2) = 0$ for some Π. Because C is finite, this set $\cap_{\Pi_0 \in C} U_{\Pi_0}$ is an (open) neighborhood of (λ_1, λ_1).

For the case $\lambda_s = \lambda_2$, there are some Π such that $\det D\Omega(\lambda_2, \lambda_2) = 0$. It follows then from standard local bifurcation theory that there exist some matrices Π and (nondegenerate) SSEG(2, 1) in the neighborhood of (λ_2, λ_2) [see Chiappori et al. (1992) for a general argument]. ■

Accordingly, the λ_1 path should be considered as the single bubble-free solution of the model, independently of the properties of the local perfect-foresight dynamics, even in the indeterminate case for this dynamics ($|\lambda_1| < |\lambda_2| < 1$). The

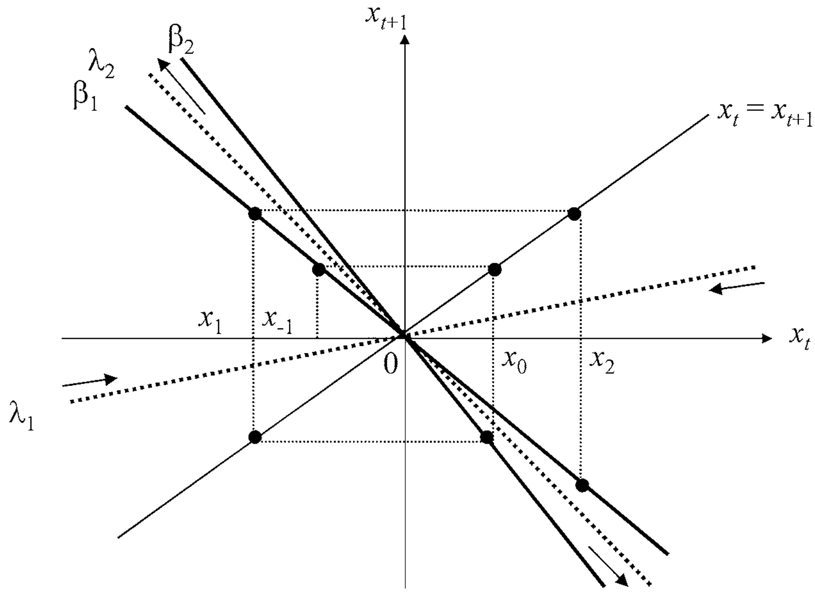

FIGURE 1. Stochastic fluctuations of the state variable induced by the sunspot equilibria, where $s_0 = s_1 = s_2 = 1$ and $s_3 = 2$.

restrictions $\gamma \neq 0$ and $\delta \neq 0$ are needed in Proposition 1. Otherwise, actual growth rates are independent of sunspot signals [see equation (5)]. However, it actually is not stringently given, first, that $\gamma \neq 0$ merely ensures that expectations matter and, second, that the bubble-free solution is easily identified in the case $\delta = 0$ (this is the steady state).

An example of stochastic fluctuations of the state variable induced by the sunspot equilibrium is depicted in Figure 1 in the hypothetical case where $s_0 = s_1 = s_2 = 1$ and $s_3 = 2$. This figure highlights the fact that the state variable is pulled out of $V(\bar{x})$ in the case $|\lambda_2| > 1$. Consequently, the stability condition $|\lambda_2| < 1$ should be met as far as we are concerned with situations in which the state variable is bounded [e.g., to ensure that it remains in $V(\bar{x})$]. It allows us to restore the conventional link between the existence of sunspot fluctuations and the indeterminacy of the stationary state that appears in models without predetermined variables [see Chiappori et al. (1992), Drugeon and Wigniolle (1994), or Shigoka (1994) among many others]. It implies, however, that fluctuations will vanish in the long run.

The next result is concerned with the issue of whether the bubble-free role of the λ_1 path is robust to a slight change in the traders' beliefs. We now precisely consider that agents randomize over the two perfect-foresight roots λ_1 and λ_2; that is, they hold beliefs (β_1, β_2) in the neighborhood of (λ_1, λ_2) [or (λ_2, λ_1)]. We show that no such beliefs are self-fulfilling as soon as the sunspot state associated with the expected growth rate β_1 near λ_1 is persistent enough, that is, π_{11} and $(1 - \pi_{22})$ are large enough. Without loss of generality, we turn our attention to SSEG(2, 1) in the neighborhood of (λ_1, λ_2) only. The case in which the SSEG(2, 1) is close to (λ_2, λ_1) would be treated in a similar way, simply by changing indexes.

PROPOSITION 2. *There exist a neighborhood of* $(\pi_{11}, \pi_{22}) = (1, 0)$ *and a neighborhood of* (λ_1, λ_2) *such that there is no SSEG*(2, 1) *in the neighborhood of* (λ_1, λ_2) *associated with a sunspot process with transition probabilities in the neighborhood of* $(\pi_{11}, \pi_{22}) = (1, 0)$.

Proof. Using the two identities, $\lambda_1\lambda_2 = \delta/\gamma$ and $\lambda_1 + \lambda_2 = -1/\gamma$, it is readily verified that the Jacobian matrix $D\Omega(\lambda_1, \lambda_2)$ of the map Ω calculated at point (λ_1, λ_2) is equal to

$$D\Omega(\lambda_1, \lambda_2) = \begin{pmatrix} \omega(\lambda_1, \lambda_2, \pi_{11})\pi_{11} - 1 & \omega(\lambda_1, \lambda_2, \pi_{11})(1 - \pi_{11}) \\ \omega(\lambda_2, \lambda_1, \pi_{22})(1 - \pi_{22}) & \omega(\lambda_2, \lambda_1, \pi_{22})\pi_{22} - 1 \end{pmatrix},$$

where $\omega(\lambda_1, \lambda_2, \pi_{ss})$ is

$$\omega(\lambda_1, \lambda_2, \pi_{ss}) = \lambda_1\lambda_2/[(1 - \pi_{ss})\lambda_1 + \pi_{ss}\lambda_2]^2.$$

The map ω is well defined when π_{ss} is in the neighborhood of 0 or 1. For $\pi_{11} = 1$ and $\pi_{22} = 0$, one has $\Omega(\lambda_1, \lambda_2) = 0$ and some computations show that det $D\Omega(\lambda_1, \lambda_2) = 1 - \lambda_1/\lambda_2$. Then, in the generic case $\lambda_1 \neq \lambda_2$, det $D\Omega(\lambda_1, \lambda_2) \neq 0$, and the implicit functions theorem applied at (λ_1, λ_2) with $\pi_{11} = 1$ and $\pi_{22} = 0$ shows that there exist neighborhoods U of (λ_1, λ_2) and V of $(\pi_{11}, \pi_{22}) = (1, 0)$ such that, for every matrix Π with transition probabilities in V, the only zero of Ω_Π in U is (λ_1, λ_2). In other words, there do not exist SSEG(2, 1) in the neighborhood of (λ_1, λ_2) associated with a matrix Π with transition probabilities in V. ■

Figure 2 gives an example of stochastic fluctuations of the state variable that are induced by the sunspot equilibrium on growth rate described in Proposition 2. Here, we set $s_0 = 1$ (so that $x_0 = \beta_1 x_{-1}$), $s_1 = 2$, and $s_2 = 1$.

A sequence of state variables sustained by some SSEG(2, 1) described in Proposition 2 remains in $V(\bar{x})$ as soon as $|\lambda_2| < 1$, that is, in the indeterminate configuration for the perfect-foresight dynamics, and it will be pulled out of $V(\bar{x})$ with probability 1 if $|\lambda_1| > 1$, that is, in the so-called source determinate configuration for these dynamics. The next result is obtained to provide a condition that ensures stability in the saddle-point case. Given the stochastic framework under consideration, the stability concept is a *statistic* criterion ensuring that, in the long run, x_t remains in the neighborhood of the steady state $\bar{x}$ with an arbitrarily high probability.

PROPOSITION 3. *Consider an SSEG*(2, 1), *denoted* $(\boldsymbol{\beta}, \Pi)$, *that sustains a sequence of stochastic realizations* $\{x_t\}_{t=-1}^{+\infty}$. *It is called "stable" if and only if, for every* $\varepsilon > 0$, *there exists a date T such that* $P\ (\forall\, t \geq T, |x_t - \bar{x}| \leq \varepsilon) \geq 1 - \varepsilon$. *Let* q_s *be the long-run probability of the signals s* $(s = 1, 2)$ *associated with the Markov transition matrix* Π. *Then an SSEG*(2, 1) *is stable if and only if* $|\beta_1^{q_1}\beta_2^{q_2}| < 1$. *If this stability condition holds true, then endogenous stochastic fluctuations of the state variable are vanishing asymptotically; that is,* $P\ (\lim x_t = \bar{x}) = 1$.

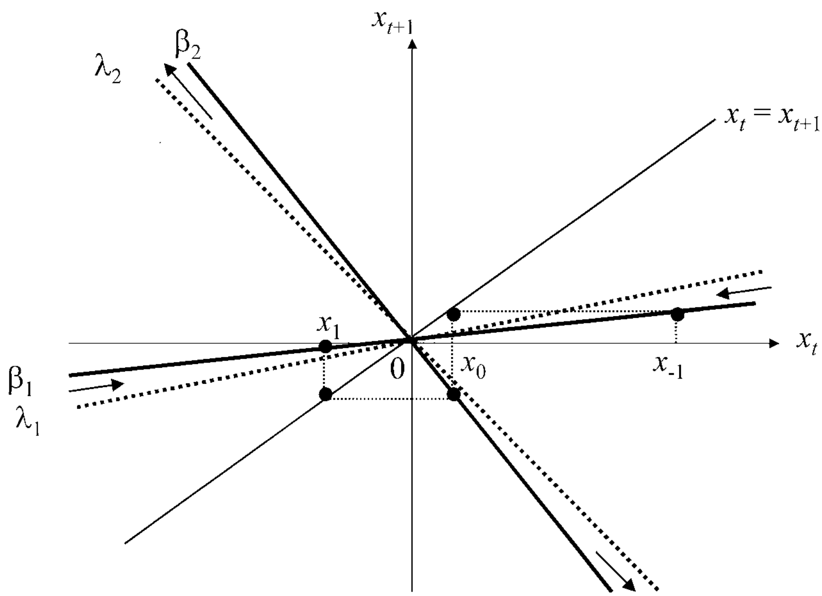

FIGURE 2. Stochastic fluctuations of the state variable that are induced by the sunspot equilibrium on the growth rate described in Proposition 2.

Proof. For the case $|\beta_1^{q_1}\beta_2^{q_2}| \neq 1$, the result comes from Theorem I.15.2 in Chung (1967). Let us consider a two-state ergodic Markov process with state space $\{\ln|\beta_1|, \ln|\beta_2|\}$ and with transition matrix Π. Applying the theorem with $f = \text{Identity}$ gives

$$P\left[\lim \frac{1}{t}\sum_{\tau=0}^{t} \ln|\beta_\tau| = q_1 \ln\beta_1 + q_2 \ln\beta_2\right] = 1.$$

Because $\ln|x_t/x_{-1}| = \sum_{\tau=0}^{t} \ln|\beta_\tau|$, one obtains

$$P\left[\lim \frac{1}{t} \ln|x_t/x_{-1}| = \ln\left|\beta_1^{q_1}\beta_2^{q_2}\right|\right] = 1.$$

If $|\beta_1^{q_1}\beta_2^{q_2}| < 1$, then $P[\lim \ln|x_t/x_{-1}| = -\infty] = 1$. Hence, $P[\lim|x_t| = 0] = 1$. Otherwise, $|\beta_1^{q_1}\beta_2^{q_2}| > 1$ and $P[\lim \ln|x_t/x_{-1}| = +\infty] = 1$. Now, $P[\lim|x_t| = +\infty] = 1$. For the case $\beta_1^{q_1}\beta_2^{q_2} = 1$, the result follows from the central limit theorem for Markov chains [Chung (1967) Theorem 1.16.1]. Let us consider the stochastic variable Y_n defined by $Y_n = \sum_{t_n \le t < t_{n+1}} \ln|\beta_s|$, where t_n is the date of the nth return to the state $\ln|\beta_1|$. The condition $q_1 \ln\beta_1 + q_2 \ln\beta_2 = 0$ implies $E(Y_n) = 0$. As $E(Y_n^2)$ differs from zero, Theorem I.14.7 in Chung (1967) is applied to get the asymptotic property

$$\lim_{t \to +\infty} P\left(\frac{1}{t}\sum_{\tau=0}^{t} \ln|\beta_\tau| > \sqrt{Bt}\right) > 0,$$

where the constant $B = q_1 E(Y_n^2)$ is independent of n according to Section I.15 in Chung (1967). Considering again $\ln|x_t/x_{-1}| = \sum_{\tau=0}^{t} \ln|\beta_\tau|$ proves the result. ■

3. GENERAL FRAMEWORK

We now deal with general economies where the current state depends on the (common) forecast of the next state and also on $L \geq 1$ predetermined variables through the following map:

$$\gamma E(x_{t+1} \mid I_t) + x_t + \sum_{l=1}^{L} \delta_l x_{t-l} = 0, \tag{7}$$

where parameter δ_l $(1 \leq l \leq L)$ represents the relative contribution to x_t of the predetermined state of period $t - l$. The dynamics with perfect foresight now involve $(L+1)$ perfect-foresight roots $\lambda_1, \ldots, \lambda_{L+1}$ (with $|\lambda_1| < \cdots < |\lambda_{L+1}|$). We shall concentrate attention on equilibrium paths along which the number of lags that influence the current state is equal to the number L of predetermined variables, that is, paths defined by L coefficients only. As a consequence such paths have *a priori* no special characteristics that would justify the label bubble-free. Then, the issue is whether the path corresponding to the L perfect-foresight roots of lowest modulus $\lambda_1, \ldots, \lambda_L$ (that is the one that corresponds to the saddle stable path in the saddle-point case $|\lambda_L| < 1 < |\lambda_{L+1}|$) still deserves to be considered as the unique bubble-free solution. According to McCallum's (1999) MSV criterion, this is the case because it is the only solution that always displays a minimal number of lags, even in the degenerate case $\delta_1 = \cdots = \delta_L = 0$ (this path then reduces to the steady state $x_t = \bar{x}$). To answer this question as we did in the preceding section, we build sunspot equilibria over L-dimensional vectors whose components stand arbitrarily close to the L coefficients that define each path with L lags. It turns out that the solution corresponding to the L perfect-foresight roots of lowest modulus is the unique solution that has no sunspot equilibrium in its neighborhood, independently of the properties of the local dynamics with perfect foresight.

3.1. Deterministic Rational Expectations Equilibria

The state variable perfect-foresight dynamics in $V(\bar{x})$ are related to the $(L+1)$ perfect-foresight roots $\lambda_1, \ldots, \lambda_{L+1}$ of the characteristic polynomial

$$P_x(x) = \gamma x^{L+1} + x^L + \sum_{l=1}^{L} \delta_l x^{L-l},$$

corresponding to (7) under the perfect-foresight hypothesis $E(x_{t+1} \mid I_t) = x_{t+1}$, namely,

$$\gamma x_{t+1} + x_t + \sum_{l=1}^{L} \delta_l x_{t-l} = 0. \tag{8}$$

We assume again that the roots of P_x are real, with $|\lambda_1| < \cdots < |\lambda_{L+1}|$. A local perfect-foresight equilibrium is a sequence of state variables $\{x_t\}_{t=-L}^{\infty}$ associated with the initial condition $(x_{-1}, \dots, x_{-L}) \in V(\bar{x}) \times \cdots \times V(\bar{x})$ and such that (8) holds at each period. Solutions where the current state depends on $(L+1)$ lags in equilibrium, namely,

$$x_t = -(1/\gamma)x_{t-1} - \sum_{l=1}^{L}(\delta_l/\gamma)x_{t-1-l},$$

are bubble solutions since beliefs matter at date $t = 0$. In what follows, we focus on solutions with only L lags. They are such that when traders hold for sure that the law of motion,

$$x_t = \sum_{l=1}^{L} \beta_l x_{t-l}, \tag{9}$$

governs the state variable behavior for every x_{t-l} $(l = 1, \dots, L)$ in $V(\bar{x})$ and every t, and when traders consequently form their forecasts, that is,

$$E(x_{t+1} \mid I_t) = \sum_{l=1}^{L} \beta_l x_{t+1-l}, \tag{10}$$

then the actual dynamics make their initial guess self-fulfilling. These actual dynamics occur once (10) is reintroduced into (7):

$$x_t = -\sum_{l=1}^{L}[(\delta_l + \gamma\beta_{l+1})/(1+\gamma\beta_1)]\, x_{t-l}, \tag{11}$$

with the convention that $\beta_{L+1} = 0$. Then, beliefs (9) are self-fulfilling whenever (9) and (11) coincide; that is,

$$\beta_l = -(\delta_l + \gamma\beta_{l+1})/(1+\gamma\beta_1) \tag{12}$$

for $l = 1, \dots, L$. Solutions of (12) will be called stationary *extended growth rates* [henceforth, stationary EGR(L)], and denoted $\hat{\beta}^b = (\hat{\beta}_1^b, \dots, \hat{\beta}_L^b)$ with the convention that $\hat{\beta}^b$ governs the perfect-foresight dynamics restricted to the L-dimensional eigenspace corresponding to all the perfect-foresight roots but λ_b $(b = 1, \dots, L+1)$. The expression of stationary EGR(L) is given by Gauthier (1999). For the sake of completeness, it is restated in the next Lemma.

LEMMA 1. *Assume that the characteristic polynomial P_x corresponding to the $(L+1)$th order difference equation* (8) *admits $(L+1)$ real and distinct roots λ_b, $1 \le b \le L+1$. Let the $(L+1)$-dimensional eigenvector u_b, $1 \le b \le L+1$, be associated with λ_b. Finally, let W_b, $1 \le b \le L+1$, be the L-dimensional eigensubspace spanned by all the eigenvectors except u_b. The perfect-foresight dynamics of the state variable restricted to W_b are written as*

$$x_t = \sum_{l=1}^{L} \hat{\beta}_l^b x_{t-l},$$

where the lth entry $\hat{\beta}_l^b$ of the stationary EGR(L) $\hat{\boldsymbol{\beta}}^b$ is

$$\hat{\beta}_l^b = (-1)^{l+1} \sum_{1 \le j_1 < \cdots < j_l \le L+1} (\lambda_{j_1} \cdots \lambda_{j_l}) \quad \textit{for all} \quad j_z \neq b, z = 1, \ldots, l.$$

Proof. We first transform the dynamics (8) into a vector first-order difference equation,

$$\boldsymbol{x}_{t+1} = \boldsymbol{T}\boldsymbol{x}_t,$$

where $\boldsymbol{T}$ is the companion matrix associated with P_x and $\boldsymbol{x}_t \equiv (x_t, \ldots, x_{t-L})^T$ (the symbol T represents the transpose of the vector). One can easily check that the $(L+1)$ eigenvalues of the $(L+1)$-dimensional matrix $\boldsymbol{T}$ are the perfect-foresight roots λ_b, $1 \le b \le (L+1)$, and that each λ_b is associated with the $(L+1)$-dimensional eigenvector $\boldsymbol{u}_b$,

$$\boldsymbol{u}_b \equiv \left(\lambda_b^L, \lambda_b^{L-1}, \ldots, 1\right)^T.$$

For every b, the perfect-foresight trajectory that is restricted to W_b is such that x_t is a linear combination of all the $\boldsymbol{u}_{b'}$ except $\boldsymbol{u}_b$; that is, $\det(x_t, P_{-b}) = 0$ where P_{-b} is the $(L+1) \times L$ matrix whose columns are all the $\boldsymbol{u}_{b'}$ except $\boldsymbol{u}_b$. Developing the determinant, this latter identity can be rewritten as

$$x_t = \sum_{l=1}^{L} a_l x_{t-l},$$

where each coefficient a_l is $(-1)^{l+1}\Delta_l/\Delta_0$ and the Δ_l are minors of the $(L+1)$-dimensional matrix (x_t, P_{-b}). Notice [see Arnaudiès and Fraysse (1987)] that Δ_0 is the determinant of Vandermonde and $\Delta_l = \sigma_l(\boldsymbol{\lambda}_{-b})\Delta_0$ where $\sigma_l(\boldsymbol{\lambda}_{-b})$ is the lth elementary symmetric polynomial evaluated at $\boldsymbol{\lambda}_{-b}$ (the L-dimensional vector whose components are all the perfect-foresight roots except λ_b):

$$\sigma_l(\boldsymbol{\lambda}_{-b}) = \sum_{\substack{1 \le j_1 < \cdots < j_l \le L+1 \\ j_l \neq b}} \lambda_{j_1}\lambda_{j_2} \cdots \lambda_{j_l}. \qquad \textbf{(13)}$$

The result follows. ■

There are $(L+1)$ stationary EGR(L) associated with $(L+1)$ different L-dimensional eigensubspaces of the $(L+1)$-dimensional local perfect-foresight dynamics (8). We now study whether the $\hat{\boldsymbol{\beta}}^{L+1}$ path is still the unique bubble-free solution by constructing sunspot equilibria over L-dimensional vectors that stand arbitrarily close to each stationary EGR(L) of the economy. This $\hat{\boldsymbol{\beta}}^{L+1}$ path is associated with the L-dimensional eigenspace corresponding to all the perfect-foresight roots except λ_{L+1}, and it governs the saddle stable path in the so-called saddle-point configuration for the perfect-foresight dynamics ($|\lambda_L| < 1 < |\lambda_{L+1}|$).

3.2. Stochastic Sunspot Rational Expectations Equilibria

Consider that agents observe a k-state discrete-time Markov process associated with a k-dimensional stochastic matrix Π. When the signal is s at the outset of period t, that is, $s_t = s$ $(s = 1, \ldots, k)$, agents believe that the current state is linked to the L previous states according to the following law of motion:

$$x_t = \sum_{l=1}^{L} \beta_l^s x_{t-l}. \tag{14}$$

In other words, they believe that the current extended growth rate $\boldsymbol{\beta}(t) = (\beta_1(t), \ldots, \beta_L(t))$ is equal to some L-dimensional vector $\boldsymbol{\beta}^s = (\beta_1^s, \ldots, \beta_L^s)$, and they deduce from the occurrence of signal s that the next extended growth rate $\boldsymbol{\beta}(t+1)$ will be equal to $\boldsymbol{\beta}^{s'}$ $(s' = 1, \ldots, k)$ with probability $\pi_{ss'}$, where $\pi_{ss'}$ is the ss'th entry of Π. Therefore, their price expectation is written as

$$E(x_{t+1} \mid I_t) = \sum_{s'=1}^{k} \pi_{ss'} \sum_{l=1}^{L} \beta_l^{s'} x_{t+1-l} = \sum_{l=1}^{L} \sum_{s'=1}^{k} \pi_{ss'} \beta_l^{s'} x_{t+1-l} \equiv \sum_{l=1}^{L} \bar{\beta}_l^s x_{t+1-l},$$

where $\bar{\beta}_l^s$ represents the average weight of x_{t+1-l} in the forecast rule when $s_t = s$. The information set I_t must accordingly be formed by the current sunspot signal $s_t = s$ and the L previous realizations x_{t-l} $(l = 1, \ldots, L)$. The actual dynamics in state $s_t = s$ are obtained by reintroducing forecasts into the temporary equilibrium map. With the convention that $\beta_{L+1}^s = 0$, one gets

$$\gamma \sum_{l=1}^{L} \bar{\beta}_l^s x_{t+1-l} + x_t + \sum_{l=1}^{L} \delta_l x_{t-l} = 0$$

$$\Leftrightarrow x_t = -\sum_{l=1}^{L} \left[\left(\gamma \bar{\beta}_{l+1}^s + \delta_l\right) / \left(\gamma \bar{\beta}_1^s + 1\right)\right] x_{t-l} \equiv \sum_{l=1}^{L} \Omega_l\left(\bar{\beta}_1^s, \bar{\beta}_{l+1}^s\right) x_{t-l}. \tag{15}$$

DEFINITION 2. *A SSEG(k, L) is a kL-dimensional vector $\boldsymbol{\beta} = (\boldsymbol{\beta}^1, \ldots, \boldsymbol{\beta}^k)$ where $\boldsymbol{\beta}^s$ is an L-dimensional vector $(\beta_1^s, \ldots, \beta_L^s)$, and a k-dimensional stochastic matrix Π such that (i) there are s and s' such that $\boldsymbol{\beta}^s \neq \boldsymbol{\beta}^{s'}$, and (ii) $\beta_l^s =$*

$\Omega_l(\bar{\beta}_1^s, \bar{\beta}_{l+1}^s)$ *for* $l = 1, \ldots, L$ *and* $s = 1, \ldots, k$, *with the convention that* $\beta_{L+1}^s = 0$ *for every* s.

An SSEG(k, L) is, accordingly, a k-state sunspot equilibrium over EGR(L). This is a situation in which every initial guess β_l^s in (14) coincides with the actual realization $\Omega_l(\bar{\beta}_1^s, \bar{\beta}_{l+1}^s)$ in (15), whatever the current sunspot signal s is; that is, beliefs about EGR(L) are self-fulfilling. The $(L+1)$ stationary EGR(L) may be called degenerate SSEG(k, L) because, for any Π, only condition (i) fails to hold true in the Definition 2.

We first consider local stochastic fluctuations in the immediate vicinity of every given stationary EGR(L). As in the two-sunspot state case, we say that a neighborhood of an SSEG(k, L) denoted $(\boldsymbol{\beta}, \Pi)$ is a product set $V \times \mathcal{M}_k$, where V is a neighborhood of the vector $\boldsymbol{\beta}$ in $\mathbf{R}^{kL}$ and $\mathcal{M}_k$ is the set of all k-dimensional stochastic matrices Π. Hence an SSEG(k, L) denoted $(\boldsymbol{\beta}, \Pi)$ is in the neighborhood of an EGR(L) $\hat{\boldsymbol{\beta}}^b$ whenever $\boldsymbol{\beta}$ stands close enough to the kL-dimensional vector $(\hat{\boldsymbol{\beta}}^b, \ldots, \hat{\boldsymbol{\beta}}^b)$. The next result extends Proposition 1.

PROPOSITION 4. *Consider the reduced form* (7). *Assume that* $\gamma \neq 0$, *that is, expectations matter*, *and* $\delta_L \neq 0$. *Then there exist SSEG*(k, L) *is every neighborhood of the stationary EGR*(L) $\hat{\boldsymbol{\beta}}^b$ *for any* $b \neq L+1$. *On the contrary*, *there is a neighborhood of stationary EGR*(L) $\hat{\boldsymbol{\beta}}^{L+1}$ (*governing perfect-foresight dynamics restricted to the eigensubspace corresponding to the L perfect-foresight roots of lowest modulus*) *in which there do not exist any SSEG*(k, L).

Proof. Let $\bar{\boldsymbol{\beta}}$ denote the kL-dimensional vector $(\bar{\beta}_1^1, \bar{\beta}_2^1, \ldots, \bar{\beta}_L^1, \bar{\beta}_1^2, \ldots, \bar{\beta}_L^2, \bar{\beta}_1^k, \ldots, \bar{\beta}_L^k)$. Then, linearizing the equilibrium condition $\Omega_l(\bar{\beta}_1^s, \bar{\beta}_{l+1}^s) = \beta_l^s$ $(l = 1, \ldots, L$ and $s = 1, \ldots, k)$ in the neighborhood of the kL-dimensional vector $(\hat{\boldsymbol{\beta}}^b, \ldots, \hat{\boldsymbol{\beta}}^b)$ leads to [notice that $\bar{\boldsymbol{\beta}}$ reduces to $(\hat{\boldsymbol{\beta}}^b, \ldots, \hat{\boldsymbol{\beta}}^b)$ at the point $(\hat{\boldsymbol{\beta}}^b, \ldots, \hat{\boldsymbol{\beta}}^b)$]

$$\boldsymbol{\beta} = F\bar{\boldsymbol{\beta}}, \tag{16}$$

where F is a kL-dimensional matrix equal to

$$F = \begin{pmatrix} F(\hat{\boldsymbol{\beta}}^b) & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & F(\hat{\boldsymbol{\beta}}^b) \end{pmatrix},$$

where

$$F(\hat{\boldsymbol{\beta}}^b) = -\frac{\gamma}{\gamma\bar{\beta}_1^b + 1}\begin{pmatrix} \hat{\beta}_1^b & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ \vdots & 0 & \cdots & 0 & 1 \\ \hat{\beta}_L^b & 0 & \cdots & \cdots & 0 \end{pmatrix},$$

with 0 the L-dimensional zero matrix. It is shown by Gauthier (1999) that the L eigenvalues of the L-dimensional matrix $F(\hat{\beta}^b)$ are λ_j/λ_b for every $j \neq b$ ($j, b = 1, \dots, L+1$). Observe now that $\bar{\beta} = (\Pi \otimes I_L)\beta$ where the symbol $\otimes$ represents the Kronecker product, and where I_L is the L-dimensional identity matrix. Remark also that $F = I_L \otimes F(\hat{\beta}^b)$. As a result, (16) becomes

$$\beta = \big(I_L \otimes F(\hat{\beta}^b)\big)(\Pi \otimes I_L)\beta \Leftrightarrow \beta = (\Pi \otimes F(\hat{\beta}^b))\beta$$

$$\Leftrightarrow \big[I_{kL} - (\Pi \otimes F(\hat{\beta}^b))\big]\beta = 0.$$

Since $(\hat{\beta}^b, \dots, \hat{\beta}^b)$ is a solution of this system, the same argument as the one used in the proof of Proposition 1 shows that there exist SSEG(k, L) in the neighborhood of $(\hat{\beta}^b, \dots, \hat{\beta}^b)$ if and only if

$$\det\big[I_{kL} - (\Pi \otimes F(\hat{\beta}^b))\big] = 0. \tag{17}$$

Let μ_s ($s = 1, \dots, k$) be an eigenvalue of Π. Then, the eigenvalues of $I_{kL} - (\Pi \otimes F(\hat{\beta}^b))$ are of the form $1 - \mu_s\lambda_j/\lambda_b$ for $s = 1, \dots, k$ and $j = 1, \dots, L+1$ and $j \neq b$ [see Magnus and Neudecker (1988)]. So that (17) admits a solution Π if and only if there exists λ_j, $j \neq b$, such that λ_b/λ_j is an eigenvalue of Π. Therefore, given that $|\mu_s| \leq 1$ and $|\lambda_{L+1}|$ is the root of largest modulus, (17) is satisfied for some Π if and only if $b \neq L+1$. ■

Hence our approach fits McCallum's conjecture in the general framework considered in this section in the sense that the equilibrium path defined by $\hat{\beta}^{L+1}$ is the only one that is free of any sunspot equilibria in its neighborhood. This result builds upon that of Gauthier (1999) who provides related arguments for the selection of the solution corresponding to the L roots of lowest modulus. Gauthier (1999) actually shows that this bubble-free path is the only one that is locally determinate in perfect-foresight dynamics on extended growth rates. Although Proposition 4 is independent of the stability (determinacy) properties of the local perfect-foresight dynamics, attention should be focused only on the indeterminate configuration ($|\lambda_{L+1}| < 1$) for these dynamics, as long as one prevents the state variable from leaving $V(\bar{x})$.

Example

Figure 3 gives an example of such sunspot equilibria. It actually represents subspaces that trigger the law of motion of the state variable in $V(\bar{x})$ in the case $L = 2$; that is, the perfect-foresight dynamics are governed by three perfect-foresight roots: λ_1, λ_2 and λ_3 (with $|\lambda_1| < |\lambda_2| < |\lambda_3|$). The two-dimensional subspace W_2 is spanned by eigenvectors associated with λ_1 and λ_3. As shown in Lemma 1, the dynamics restricted to W_2 are $x_t = (\lambda_1 + \lambda_3)x_{t-1} - \lambda_1\lambda_3 x_{t-2}$. It follows from Proposition 4 that it is possible to build SSEG(k, 2) close to W_2. Here, $k = 2$, so that these equilibria are defined by the same two-dimensional stochastic matrix Π and two different two-dimensional vectors (β_1^s, β_2^s) for $s = 1, 2$. Both vectors stand arbitrarily close to $(\lambda_1 + \lambda_3, -\lambda_1\lambda_3)$. They define the two-dimensional subspaces

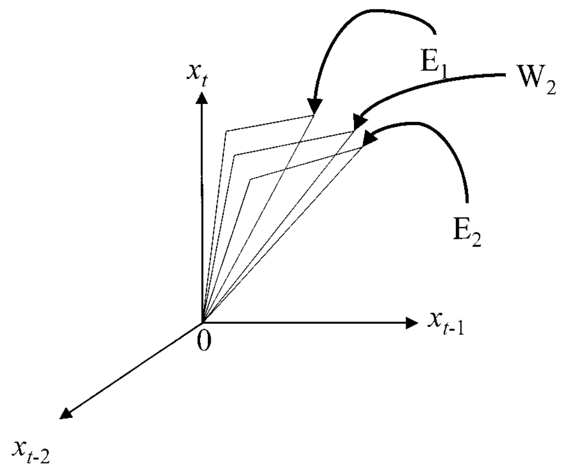

FIGURE 3. Example of sunspot equilibria in which subspaces trigger the law of motion of the state variable in $V(\bar{x})$ in the case $L = 2$.

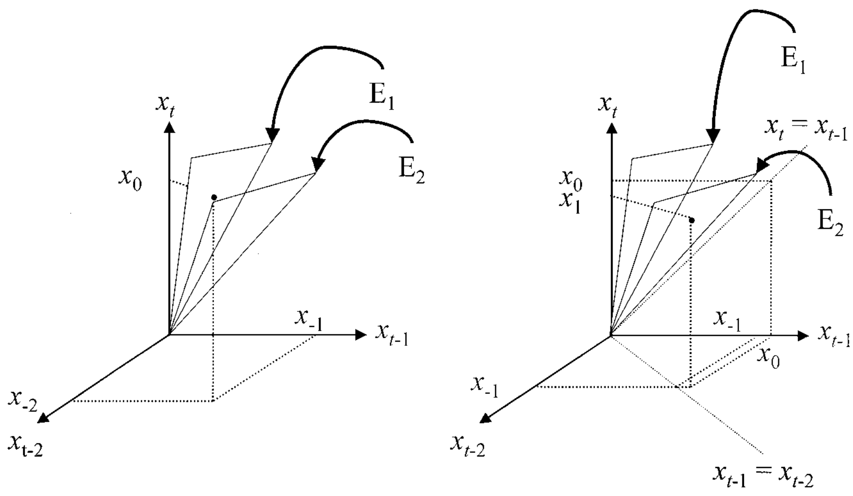

FIGURE 4. Change in the value of the state variable for $s_0 = 1$ and $s_1 = 2$.

E_1 and E_2, respectively. The state variable will alternate between E_1 and E_2 according to the current sunspot signal. In Figure 4, we depict the change in the value of the state variable for $s_0 = 1$ and $s_1 = 2$.

As in our preliminary example, we now ask whether the bubble-free role of the path defined by $\hat{\beta}^{L+1}$ will be maintained in the case in which agents randomize over different stationary EGR(L). For simplicity, we assume that $k = L + 1$, i.e., all the stationary EGR(L) enter the support of the beliefs. Precisely, we consider that traders hold beliefs β in the neighborhood of $(\hat{\beta}^1, \ldots, \hat{\beta}^{L+1})$. The next result extends Proposition 2: We show that there is no SSEG($L + 1, L$) as soon as the sunspot state associated with the expected growth rate β^{L+1} near $\hat{\beta}^{L+1}$ is persistent enough, that is, every $\pi_{s(L+1)}$ is large enough.

PROPOSITION 5. *There exists a neighborhood of* $(\pi_{1(L+1)}, \dots, \pi_{(L+1)(L+1)}) = (1, \dots, 1)$ *and a neighborhood of* $(\hat{\beta}^1, \dots, \hat{\beta}^{L+1})$ *such that there is no SSEG*$(L+1, L)$ *in the neighborhood of* $(\hat{\beta}^1, \dots, \hat{\beta}^{L+1})$ *associated with a sunspot process with transition probabilities in the neighborhood of* $(\pi_{1(L+1)}, \dots, \pi_{(L+1)(L+1)}) = (1, \dots, 1)$.

Proof. The proof mimics the proof of Proposition 4. Consider the following $L(L+1)$-dimensional matrix:

$$G = \begin{pmatrix} F(\hat{\beta}^1) & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & F(\hat{\beta}^{L+1}) \end{pmatrix},$$

where the $F(\hat{\beta}^b)$ are the L-dimensional matrices defined in the proof of Proposition 4 (notice $\bar{\beta}_l^b$ is now different from $\hat{\beta}_l^b$). There exist SSEG$(L+1, L)$ in the neighborhood of $(\hat{\beta}^1, \dots, \hat{\beta}^{L+1})$ if and only if, for some matrix Π,

$$\det[I_{L(L+1)} - G(\Pi \otimes I_L)] = 0.$$

Notice now that

$$G(\Pi \otimes I_L) = \begin{pmatrix} \pi_{11}F(\hat{\beta}^1) & \cdots & \pi_{1L+1}F(\hat{\beta}^1) \\ \vdots & \ddots & \vdots \\ \pi_{L+11}F(\hat{\beta}^{L+1}) & \cdots & \pi_{L+1L+1}F(\hat{\beta}^{L+1}) \end{pmatrix}.$$

At the point $[\pi_{1(L+1)}, \dots, \pi_{(L+1)(L+1)}] = (1, \dots, 1)$, this matrix reduces to

$$G(\Pi \otimes I_L)_{(\pi_{1(L+1)},\dots,\pi_{(L+1)(L+1)})=(1,\dots,1)} = \begin{pmatrix} 0 & \cdots & 0 & F(\hat{\beta}^1) \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & F(\hat{\beta}^{L+1}) \end{pmatrix}.$$

It is then straightforward that the eigenvalues of this matrix are 0 with multiplicity L^2 and each eigenvalue of $F(\hat{\beta}^{L+1})$ with multiplicity 1. Some computations show that the eigenvalues of $F(\hat{\beta}^{L+1})$ are $\lambda_j / \sum_{s=1}^{L+1} \pi_{js}\lambda_s$ for $j = 1, \dots, L$. These computations are precisely as follows: Every λ_j for $j \neq L+1$ satisfies the polynomial identity $\lambda_j^L = \sum_{l=1}^{L} \hat{\beta}_l^{L+1} \lambda_j^{L-l}$. It follows that $-\lambda_j\gamma/(\gamma\bar{\beta}_1^{L+1} + 1)$ is an eigenvalue of the transpose of $F(\hat{\beta}^{L+1})$ [associated with the eigenvector $(\lambda_j^{L-1}, \dots, \lambda_j, 1)$] and it is then an eigenvalue of $F(\hat{\beta}^{L+1})$ itself. Because $\bar{\beta}_1^{L+1} = \sum_{s=1}^{L+1} \pi_{L+1s}\hat{\beta}_1^s$, $\hat{\beta}_1^s = \sum_{j\neq s} \lambda_j$, and $\sum_j \lambda_j = -1/\gamma$, it follows that $-(\gamma\bar{\beta}_1^{L+1} + 1)/\gamma = \sum_{s=1}^{L+1} \pi_{L+1s}\lambda_s$.

Finally, because the perfect-foresight roots λ_s are assumed to be distinct and smaller than λ_{L+1} in modulus, no eigenvalue of $F(\hat{\beta}^{L+1})$ is equal to 1 and no eigenvalue of $[I_{L(L+1)} - G(\Pi \otimes I_L)]$ is equal to 0 when $(\pi_{1(L+1)}, \ldots, \pi_{(L+1)(L+1)}) = (1, \ldots, 1)$. Hence, its determinant is not equal to 0 either. Then, by continuity of the determinant with the coefficients $\pi_{ss'}$, there is a compact neighborhood of the point $(\pi_{1(L+1)}, \ldots, \pi_{(L+1)(L+1)}) = (1, \ldots, 1)$ such that the determinant $\det[I_{L(L+1)} - G(\Pi \otimes I_L)]$ is nonzero for every matrix with transition probabilities in this neighborhood. Applying the same argument as the one used in proof of Proposition 1 shows the result. ■

The purpose of the next proposition is to provide a coudition that ensures stability in the case in which the stationary state is locally determinate in the perfect-foresight dynamics ($|\lambda_{L+1}| > 1$). The stability concept is the same as the one defined in Proposition 3.

PROPOSITION 6. *Consider an SSEG(k, L) defined by $(\boldsymbol{\beta}, \Pi)$ that sustains a sequence of stochastic realizations $\{x_t\}_{t=-L}^{+\infty}$. Let B_s be the L-dimensional companion matrix associated with the L-dimensional vector $\boldsymbol{\beta}^s$:*

$$B_s = \begin{pmatrix} \beta_1^s & \cdots & \cdots & \beta_L^s \\ 1 & \ddots & & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Let $\|B_s\| = \sup_{|z|=1} |B_s z|$ the norm of matrix B_s. Let q_s be the long-run probability of the signal s $(s = 1, \ldots, k)$ corresponding to Π. Then, an SSEG(k, L) is stable if $\prod_{s=1}^{k} \|B_s\|^{q_s} < 1$. If this stability condition holds true, then endogenous stochastic fluctuations of the state variable are vanishing asymptotically; that is, $P(\lim x_t = \bar{x}) = 1$.

Proof. When the current signal is s, then the L-dimensional vector $\boldsymbol{x}_t = (x_t, \ldots, x_{t-L})$ is given by

$$\boldsymbol{x}_t = B_s \boldsymbol{x}_{t-1}.$$

Hence, for an history of the sunspot process $s_0, \ldots, s_t$, one obtains

$$\boldsymbol{x}_t = B_{s_t} \cdots B_{s_0} \boldsymbol{x}_{-1}.$$

A standard result on matrix norms is

$$\|\boldsymbol{x}_t\| \leq \left\|B_{s_t}\right\| \cdots \left\|B_{s_0}\right\| \|\boldsymbol{x}_{-1}\|,$$

which is rewritten

$$\ln \frac{\|\boldsymbol{x}_t\|}{\|\boldsymbol{x}_{-1}\|} \leq \sum_{\tau=0}^{t} \ln \left\|B_{s_\tau}\right\|.$$

Consider then the k-state ergodic Markov process with state space $\{\ln \|B_1\|, \ldots, \ln \|B_k\|\}$ and with transition matrix Π. The proposition follows from Theorem I.15.2 in Chung (1967) as in the two-sunspot state case of Proposition 3. ■

4. CONCLUSION

The purpose of this paper was to provide a criterion allowing for the definition of the bubble-free solutions in dynamic rational expectations models. We have studied whether (Markovian) sunspot-like beliefs can be self-fulfilling in the neighborhood of candidates solutions for the label "bubble-free," that is, those solutions that do not display irrelevant lags with respect to the number of initial conditions. We have shown that there is only one equilibrium path close to which the sunspot fluctuations under consideration cannot arise, and we have emphasized that the choice of this path is independent of the local properties of the perfect-foresight dynamics. It is worth noticing that, as soon as the suitable dynamics with perfect foresight on (extended) growth rates is written, as done by Gauthier (1999), this existence result is in accordance with the well-known results linking the existence of sunspot equilibria to determinacy properties of the (correctly chosen) perfect-foresight dynamics. Finally, the unique bubble-free path belongs to the eigensubspace of the perfect-foresight dynamics spanned by the L roots of lowest modulus. It is the solution identified by McCallum's (1999) MSV criterion. Accordingly, it fits the conventional wisdom that the saddle stable path is the unique fundamentals solution in the saddle-point configuration.

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