Regulation of Brown and Green Firms: the case of irrational changes in lifestyles*(V.244)

Denis Claude
CNRS & Centre d'Economie de la Sorbonne

Georges Zaccour
Chair in Game Theory and Management, GERAD, HEC Montréal

February 4, 2011

Abstract

Consumers do not necessarily understand the economic trade-offs that determine the optimal level of pollution and the functioning of environmental policy instruments. When the optimal policy is implemented, market mechanisms ensure that consumption decisions are guided by correct price signals. However, green consumers may not simply react to these market signals but continue to adjust their willingness to pay for the products as pollution accumulates into the natural environment. The behaviour of green consumers then becomes a source of market failure. We consider a dynamic model of pollution control in which the environmental regulator anticipates consumers' lack of responsiveness to environmental policy implementation. In this context, a paternalist regulator will design its environmental policy so as to correct both the environmental externality problem and the behavioural market failure. By contrast, a populist regulator will correct the environmental externality problem only. We characterize and compare the tax/subsidy policies that would be chosen by both types of planners to regulate a polluting duopoly. The implementation of the populist policy may lead either to an excessive or an insufficient level of pollution accumulation. Furthermore, it may require to provide higher subsidies to the brown firm in the long-run.

1 Introduction

We consider a green market with two firms which generate pollution emissions that accumulate over time in the ambient environment. We assume that these two firms differ in their emission-output ratios. Furthermore, we suppose that consumers are environmentally conscious and prefer environmentally friendlier products. Because of this concern for environmental quality, they perceive the products offered by the two firms as vertically differentiated. Facing an industry that disregards the monetary cost of pollution damages, consumers are assumed to integrate (the totality or part of) this cost into their decision process. Specifically, consumers are willing to pay less (resp. more) for acquiring the brown (resp. green) product as the stock of pollution increases and the environmental problem becomes more severe. Then, everything else being equal, an increase in the pollution stock leads consumers to substitute away from the brown good. Consequently, if consumers' behavior is based on a sound assessment of the social costs associated with pollution emission, it is prone to mitigate the environmental problem. However, when the optimal economic instrument is used to secure the fulfillment of the social optimum, consumers should recognize that the environmental issue is being dealt with and revise their decision process accordingly. If they fail to do so, their behavior becomes in itself a source of market failure: *the continual adjustment of consumers' willingness to pay for the two goods may reduce social welfare*.

In the present paper, we assume that consumers will not react to the intervention of the environmental regulator and will continue to adjust their preferences over time, even through the optimal environmental policy is implemented. Furthermore, we suppose that the environmental regulator is able to anticipate this lack of responsiveness of the consumers. In this context, the regulator may behave paternalistically and correct consumers' preferences prior to the design of the optimal policy. Alternatively, the regulator may behave populistically, taking consumers' preferences as given and designing its environmental

*Preliminary and incomplete: please do not quote.

policy on the basis of uninformed environmental worries. However, in both cases, he or she has to take into account the evolution of consumers' willingness to pay for the product at the market stage.

The structure of the paper is as follows. The basic ingredients of the model are presented in section 3. We characterize the optimal environmental policy for a paternalist regulator in section 4. The policy that would be chosen by a populist regulator is characterized in section 5. In section 5.1, we assume that the social planner only knows the preferences of the average consumer. That assumption is relaxed in section 5.2. In section 6 we characterize and compare all the possible evolutions of the instrument as the stock of pollution increases for the special case where $\gamma = 0$. Section **??**contrasts graphically the respective evolutions of the populist and paternalist policies. Section 7 concludes.

2 Green consumers, behavioural failures and public policies

Over the last few decades, protecting and sustaining the environment as become an important policy issue in virtually all countries. The heightened coverage of environmental problems and dramatic reports of environmental degradations have produced a sense of crisis and led to a surge in public concern. In particular, consumers have grown increasingly aware of how their lifestyles and consumption patterns affect the environment. This raised awareness has partly translated into positive changes in consumption behavior towards the environment.

Today consumers place a growing interest on the environmental impact of the products and services they consume. Also, they prove to be increasingly sensitive to information regarding the environmental performance of firms. In other words, consumers are ready to differentiate between products on the basis of their environmental impacts provided that they are informed of how they fare along the environmental quality dimension. More importantly, there is substantial evidence to indicate that some of them are willing to pay more for sustainable and environmentally friendly products (energy-efficient appliances, organic food or bio-detergents, for example). In economic terms, those consumers who agree to a price premium for environmental quality deliberately internalize the negative externalities arising from consumption and production activities.

The emergence of green consumers has important implications for the supply side. From firms' perspective environmental product differentiation resembles other forms of vertical product differentiation. It allows producers to exercise some additional control over the price of their own specific products. Eventually, it provides industry rivals with the opportunity to soften product market competition, extract more of the consumers' surplus and increase their profits. The witnessed increase in the availability of "green" alternatives to standard "brown" goods is an obvious indication that firms are becoming increasingly mindful of this opportunity.

It goes without saying, however, that unscrupulous producers have an incentive to jump on this lucrative bandwagon by deceptively advertising their products as "green". This practice, known as Greewashing, may allow sly producer to increase their profits in the short-run at the expense of both consumers and reputable firms. However, in the long-run, the fraud will become apparent to consumers and undermine their confidence in green marketing so that ultimately green markets may collapse. That threat provides a clear-cut rationale for the observed emergence of green labels. However, reputable firms have alternative strategies at their disposal to signal the environmental quality of their products. In particular, price may constitute an effective channel[1].

The willingness of consumers to pay for environmental quality sounds as a good news for the environment. But, to what extent does this partial and voluntary internalization of environmental externalities can replace environmental policy ? This question is adressed by Eriksson [2004] who investigates the impact of green consumerism on market equilibrium under imperfect competition. He considers a simple model of horizontal product differentiation with two firms whose products are located at different ends of the quality scale. He shows that green consumerism fails to induce both firms to adopt clean production technologies unless consumers voluntarily internalize the negative externality entirely. Indeed, when consumers differ in their degree of environmental conciousness or internalize only part of the externality, firms have no incentive to adjust towards clean production because they would loose the benefits of environmental quality differentiation. Hence, a public intervention is required to regulate green markets.

This negative result should be put into perspective, however. As noted by Eriksson [2004], existing environmental regulations may be lax so that green consumerism might be almost as effective in coping with pollution. Furthermore, as shown by Arora and Gangopadhyay [1995], green consumerism may induce firms to overcomply with existing environmental regulations.

[1] See Mahenc [2007, 2008].

A number of papers have recently been published on the regulation of green markets (e.g., Arora and Gangopadhyay [1995], Bansal and Gangopadhyay [2003], Conrad [2005], Cremer and Thisse [1999], Eriksson [2004], Lombardini-Riipinen [2005], Mahenc [2007, 2008], Moraga-Gonzalez and Padron-Fumero [2002], Motta and Thisse [1999]). Most of that literature implicitly assumes that the concern expressed by consumers for the environment is legitimate. By contrast, Conrad [2005] suppose that the environmental regulator and consumers disagree about the environmental impact of a product. In this case, consumers' behavior becomes a source of market failure and environmental policy should aim at correcting the environmental misperception. In order to restore social efficiency, Conrad [2005] suggests that the environmental regulator should launch an information campaign to convince consumers that their environmental concern is misplaced[2].

In actual practice, however, such a campaign may fail. There are several reasons why this may be the case. First, as noted by Alba and Hutchinson [2000], "consumers are overconfident – they think they know more than they actually do". This psychological trait has important consequences. It tends to make consumers impervious to scientifically established facts and arguments that conflicts with their own beliefs and experiences. In fact, consumers want scientists to find evidence for what they see as obvious environmental problems (Walker et al. [2006]).

Second, public information campaigns may not result in a greater public understanding of complex environmental issues. A classical example is public education on global warming. Despite considerable efforts to educate the public, empirical surveys highlight a tendancy among citizens to mistake the causes of ozone layer depletion for those of global warming[3]. The following extract from a television interview by President Nicolas Sarkozy embodies a clear instance of that confusion:

> "Scientists and scholars from all over the world have met for mounths and mounths to draw up a report: the world is doomed if we continue to emit carbon that creates a hole in the ozone layer and breaks the balance of the planet" (TF1:Television interview on September 23, 2010).

Bord et al. [2000] investigate the key determinants of behavioral intentions to address global warming. They find that accurate knowledge is the strongest predictor of both stated intentions to take voluntary actions and to support new government policies to reduce greenhouse gaz emissions. By contrast, bogus knowledge (such as the belief that pesticides and aerosols contribute to climate change) contributes significantly to the belief in global warming but does not correlate significantly with government policy support. Bord et al. [2000] suggest a plausible explanation for this difference in impact on behavioral intentions. They stress that the belief in climate changes requires much less information than the choice of an appropriate course of action to mitigate global warming. Hence, people who hold bogus beliefs about the causes of global warming face a greater uncertainty as to the appropriate policy option:

> "[M]any people are not aware of the specific human activities that result in substantial carbon dioxide emissions... [T]hese respondents should be uncertain of the strategies that will be effective in reducing these emissions. Those believing that aerosols and insecticides cause global warming are not likely to make wise choices on referenda questions for governmental policies" [Bord et al., 2000]

Finally, apart from the perception of environmental issues, there are other reasons why the preferences of the regulator and that of consumers may not be aligned. Consumers may not understand environmental policy and the way it seeks to balance economic and environmental externalities. Also, they may have their suspicions about the true purposes and goals of the environmental policy implemented by the social planner. Correspondingly, they may not revise their preferences, even after the implementation of the optimal environmental policy, and continue to adjust their willingness to pay as the stock of pollution increases.

The present paper is related to Salanié and Treich [2009]'s formal analysis of the Happyville Fable[4].

[2] The role of environmental information provision as an environmental policy instrument was investigated by Petrakis et al. [2005b], Sartzetakis et al. [2009] .

[3] See, for example, Bord et al. [2000]

[4] See Portney [1992]. You are Director of Environmental Protection in Happyville (...). The drinking water supply in Happyville is contaminated by a naturally occurring substance that each and every resident believes may be responsible for the above-average cancer rate observed there. So concerned are they that they insist you put in place a very expensive treatment system to remove the contaminant. (...) The problem is this. You have asked the top ten risk assessors in the world to test the contaminant for carcinogenicity. (...). These ten risk assessors tell you that while one could never prove that the substance is harmless, they would each stake their professional reputations on its being so. You have repeatedly and skillfully communicated this to the Happyville citizenry, but because of a deep-seated skepticism of all government officials, they remain completely unconvinced and truly frightened (...).

The autors study how risk perception affect risk regulation. They consider a setting in which the citizen's beliefs about the risk related to drinking from a contaminated water supply may differ from the beliefs held by the environmental authority. In this setting, the decision to invest (or not) in a water cleanup technology proves to be controversial. Indeed, normative arguments seem to be contradictory: Consumer sovereignty arguments support the environmental regulator's decision to invest while cost-benefit arguments points to the opportunity cost involved in spending money on a phantom risk. Salanié and Treich [2009] define a populist (resp. paternalist) regulator as a regulator who maximizes the citizen's welfare computed with the citizen's beliefs (resp. his own beliefs). A natural explanation for the over-regulation of environmental risks is that environmental regulators behave as populist regulators and over-invest in clean-up technology in response to their constituents' worries. However, as shown by Salanié and Treich [2009], stringent environmental regulations may be chosen even though the environmental regulator behaves paternalistically. In other words, the observed over-regulation of environmental risks may result from the diverging risk perceptions of regulators and citizens.

3 The model

We consider a dynamic pollution control game in which a benevolent social regulator seeks to regulate a polluting duopoly that exhibits the following features.

3.1 The green market

The supply side There are two firms which compete in quantities over the infinite (and continuous) time period $[0, +\infty)$. Each firm produces one variety of an horizontally differentiated good q. We assume that the two varieties are produced through technologies that differ in their pollution intensity. We let s_i denote firm's i, $(i = b,\ g)$, constant emission/output ratio which is assumed to be fixed over the whole horizon of the game. We adopt the convention that $s_b > s_g \geq 0$. Then, it is possible to interpret q_g (resp., q_b) as the "green" (resp., the "brown") product. We introduce the notation $\Delta_i = (s_i - s_j)$ as a short-hand to denote firm i's quality advantage over firm j. By construction, we have $\Delta_g = -\Delta_b < 0$.

Pollution emissions are assumed to accumulate over time in the ambient environment, causing present as well as long-term external damages. We let $q_i(t)$ denote the quantity sold by firm i at time t. Then, assuming a constant rate of decay, the dynamics of the pollution stock $S(t)$ is given by

$$\dot{S}(t) = \left[s_g\, q_g(t) + s_b\, q_b(t)\right] - \delta\, S(t), \quad S(0) = \bar{S} \geq 0, \tag{1}$$

where the coefficient $\delta \in (0, 1)$ reflects the environment's self-cleaning capacity and $\bar{S}$ is the initial size of the pollution stock. We suppose that the economic loss resulting from the current level of the pollution stock $S(t)$ can be measured by the quadratic damage function, $D(S) = (1/2)S^2$.

Firm i's marginal cost of production is assumed to be constant and equal to $k_i \geq 0$. We let $C_i(q_i) = k_i\, q_i$ denotes Firm i's total cost function. Finally, we suppose that the marginal cost of production of the brown product is less or equal than the marginal cost of the green product; i.e., $k_g \geq k_b$.

The demand side There is a continuum of consumers of mass 1 with perfect information about firms' pollution intensity. Consumers have a taste for variety; i.e., they are willing to buy both goods. Furthermore, they perceive the two products as not only horizontally but also vertically differentiated. Here, horizontal differentiation stems from differences in the physical characteristics of the two products (color, size or shape, for example) whereas vertical differentiation originates from consumers' preference for environmentally friendly products. While all consumers have a preference for the green product, they differ in their degree of environmental consciousness, θ. Henceforth, we assume that θ is uniformly distributed across consumers on the interval $[\theta_1, \theta_2]$ with $\theta_2 = \theta_1 + 1$ so that each consumers is identified by its degree of environmental consciousness. Furthermore, the distribution of consumers and its support are supposed to be invariant with respect to time.

The utility a consumer derives from the purchase of a product i, $(i = b, g)$, depends on his/her type θ, the environmental quality differential measured by Δ_i and the current size of the pollution stock $S(t)$. This is captured by assuming that the utility function for the θ-type consumer is given by[5] :

[5]In a static setting, a similar utility function has been used in many works such as Manasakis et al. [2007] and Petrakis et al. [2005a]. This is an augmented version of the utility function introduced by Dixit [1979] and used in Singh and Vives [1984] which incorporates aspects of vertical product differentiation along the line of Hackner [2000].

$$U(\theta,\,t) \;=\; \left(a-\theta\Delta_g S(t)\right) x_g(\theta,t) + \left(a-\theta\Delta_b S(t)\right) x_b(\theta,t) - \frac{1}{2}\left(x_g^2(\theta,t)+x_b^2(\theta,t)+2\gamma x_g(\theta,t)x_b(\theta,t)\right) + m, \quad (2)$$

where $x_i(\theta,t)$ represents the quantity of good i purchased by the consumer of type θ at time t, and m is a numeraire good produced by a competitive sector. Note that the utility function (2) is quadratic in the consumption of the duopolists' products and linear in the consumption of the numeraire good. Under these assumptions there are no incomes effects in the duopolistic market and a partial equilibrium analysis can be carried out. Then, firm i's inverse demand function is given by :

$$p_i(t) = a - \bar{\theta}\,\Delta_i\,S(t) - q_i(t) - \gamma\,q_j(t). \quad (3)$$

where $\bar{\theta} = (\theta_1+\theta_2)/2$ characterizes the average-type consumer[6]. To begin with, note that the parameter γ measures the degree of horizontal product differentiation; that is, the sensitivity of consumer θ demand for product i with respect to a change in the price of product j. As $\gamma \to 1^-$, we obtain as a limit case a duopoly market with perfectly substitutable goods. By contrast, setting $\gamma = 0$ yields independent goods. In the remainder of this paper, we will restrict attention to the case of imperfectly substitutable goods and thus assume that $0 \le \gamma < 1$.

Let us now turn to vertical product differentiation. From equation (3) and, for a given pair of prices, one clearly sees that consumer θ adjusts its willingness to pay for product i as the size of the pollution stock increases and the environmental problem becomes more severe. Specifically, his/her willingness to pay for the brown (green) product decreases (increases). Hence, as a result of pollution accumulation, consumers substitute away from the pollution intensive good.

The unregulated duopoly We now characterize the behavior of the firms in the duopoly subgame. Under the above assumptions, firm i's instantaneous profit level is $\pi_i(t) = \left(a - \bar{\theta}\,\Delta_i\,S(t) - q_i(t) - \gamma\,q_j(t)\right)\,q_i(t) - k_i\,q_i(t)$, $i = b, g$. In the duopoly subgame each firm is assumed to choose its output strategy so as to maximize its long-run profit, defined as the integral of its stream of discounted short-term profits

$$\begin{aligned} \max_{q_i(.)} \quad \Pi_i \quad &= \int_0^\infty \pi_i(t)\,e^{-rt}\,dt, \quad i = b, g. \qquad (5)\\ s.t.\,(1), \qquad & q_i(t) \ge 0,\ \forall t \in [0,\infty), \end{aligned}$$

where r denotes the discount rate. The specific set of strategies that are available to the firms depend on the information structure of the game. In this paper, we assume that the two firms are able to observe the current state of the game and use this information to revise their strategies at each point of time. Consequently, each firm i ($i = b, g$) is assumed to use a markovian strategy; i.e., a decision rules of the form $q_i(t) = \Phi_i\,(S(t))$. The relevant equilibrium concept for the analysis is the Markov-perfect Nash equilibrium (MPNE). Let us recall that a MPNE is defined as a profile of markovian strategies that are mutual best responses.

3.2 Paternalist vs. populist planners

Social welfare is defined as the sum of consumers' and producers' surplus less environmental damages. However, the correct measure of social welfare depends on whether or not the social planner implements a corrective tax policy. We have assumed that consumers' preferences evolve over time as pollution accumulates into the ambient environment. More specifically, consumers reduce their willingness to pay for the brown product, and increase their willingness to pay for the green product, as the stock of pollution increases. Such a behavior may be legitimate if the social planner fails to provide a proper level of environmental quality, in which case it is prone to remedy (part of) the environmental problem. However, it becomes a source of market failure in itself when the social planner enforce the optimal environmental policy. By definition, the optimal environmental instrument achieves the right balance

[6]See Annexe A. Furthermore, observe that the aggregate demand functions (3) are indistinguishable from those of the average-type consumer $\bar{\theta}$ with utility given by:

$$U_s(t) = \left(a - \bar{\theta}\,\Delta_g\,S(t)\right)\,q_g(t) + \left(a - \bar{\theta}\,\Delta_b\,S(t)\right)\,q_b(t) - (1/2)\,\left(q_g^2(t)+q_b^2(t)+2\gamma q_g(t)q_b(t)\right) + m. \quad (4)$$

Correspondingly, the above demand system could have been derived as the solution to the decision problem of a single representative consumer located at $\bar{\theta}$. However this should not be taken as implying that the distribution of consumers is irrelevant in our model. As will be shown below, the average-type consumer's assessment of social welfare differs from that of the group of all consumers.

between economic and environmental objectives. Therefore, when it is implemented, consumers should acknowledge that the issue has been dealt with and revise their preferences accordingly, i.e., all consumers should adopt the behavior of the consumer of type $\theta = 0$.

In this paper, we assume that consumers do not revise their preferences. This remark lead us to distinguish between two different measures of social welfare: informed and uninformed social welfare. We shall denote instantaneous social welfare measures by $w_l(t) = CS_l(t) + PS_l(t) - D(S(t))$, where $l = o,\, s$ indicates whether we consider the informed (o) or the uninformed welfare measure (s). Correspondingly, agregate welfare measures will be denoted by:

$$W_l = \int_0^{+\infty} w_l(t)\, e^{-r\,t} dt, \quad l = o,\, s. \tag{6}$$

Informed social welfare Informed social welfare becomes the proper measure of social welfare as soon as the social planner intervene to regulate the market. Since the environmental policy instrument solves the environmental problem, the environmental concern expressed by the consumers is no longer legitimate. Accordingly, it should not be taken into account when measuring the actual level of social welfare. Hence, all consumers should be treated as identical and of type $\theta = 0$ for social computations. Their preferences are then described by the following utility function

$$U_o(t) = a\left(q_g(t) + q_b(t)\right) - (1/2)\left(q_g^2(t) + q_b^2(t) + 2\gamma q_g(t) q_b(t)\right) + m \tag{7}$$

which yield the standard Dixit-Singh-Vives demand system $p_i^o(t) = a - q_i(t) - \gamma\, q_j(t), \quad i = b, g$. Finally, it is straightforward to show that consumers' surplus is then given by $CS_o(t) = \frac{1}{2}(q_g(t)^2 + q_b(t)^2 + 2\,\gamma\, q_g(t)\, q_b(t))$ and informed social welfare writes as

$$w_o(t) = U^o(t) - \sum_{i=b,g} k_i\, q_i(t) - D(S(t)). \tag{8}$$

Correspondingly, we define a paternalist regulator as a regulator who maximize informed social welfare; that is, social welfare as computed with consumers' informed preferences.

Uninformed social welfare In this paper, we assume that the behavior of consumers does not respond to changes in environmental policies. Specifically, we assume that consumers fail to revise their preferences after the implementation of the optimal environmental policy. Namely, their utility functions remain given by Equation (2). In Appendix B, it is shown that aggregate consumers' surplus computed from uninformed consumers' preferences is given by:

$$CS_s(t) = CS_o(t) + \Omega\left(S\left(t\right)\right), \quad \text{with} \quad \Omega\left(S\left(t\right)\right) = \frac{\Delta_g^2}{12\left(1-\gamma\right)}\, S(t)^2. \tag{9}$$

It is important to note that $\Omega(S(t)) > 0$ and $\Omega'(S(t)) > 0$. In other words, uninformed aggregate consumers' surplus increases as the stock of pollution increases. This is due to the variability of consumers' characteristics and the fact that consumers' satisfaction increases as vertical product differentiation increases. Finally, it is straightforward to show that uninformed social welfare writes as:

$$w_s(t) = U_s(t) - \sum_{i=b,g} k_i\, q_i(t) - D(S(t)) + \Omega\left(S\left(t\right)\right). \tag{10}$$

We define a populist regulator as a regulator who maximizes uninformed social welfare; that is, social welfare as computed with consumers' uninformed preferences given by Equation (2) .

3.3 Efficiency inducing taxation

We assume that the benevolent social planner wishes to implement the social optimum. In our context, the social optimum is defined as the couple of production paths which maximizes the current value of the discounted stream of social welfare subject to the law of evolution of the pollution stock. Formally, it is obtained as the solution of the following program:

$$\max_{q_g(.),q_b(.)\geq 0} \quad W_l = \int_0^{+\infty} w_l(t)\, e^{-rt}\, dt. \tag{11}$$
$$s.t. \quad \dot{S}(t) = \left[s_g\, q_g(t) + s_b\, q_b(t)\right] - \delta S(t), \quad S(0) = \bar{S} \geq 0.$$

However, we suppose that the social planner cannot enforce this solution directly. Rather, he seeks to design a policy instrument which achieves this outcome through the decentralized market mechanism.

In the remainder of this paper, we suppose that the social planner uses a Markovian output tax (or subsidy) policy in order to decentralize the social optimum. With this purpose in mind, he/she designs and implements a system of tax (or subsidy[7]) rules $\{\tau_i(S)\}_{i=b,g}$ which condition the instantaneous rate of taxation (or subsidization) on the current level of the pollution stock. By restricting our attention to stationary Markovian tax/subsidy rules we can avoid the time inconsistency problem that would arise with time-dependent price-based policies. Indeed, as pointed out by Karp and Livernois [1992], any policy rules τ_i that depends explicitly on calendar time ($\tau_i(t)$ or $\tau_i(S,\, t)$, for example) is subject to strategic manipulations by the duopolists. By contrast, the policy scheme we consider is immune to such manipulations since it satisfies *Subgame Perfectness* (See Benchekroun and Van Long [1998]; Karp and Livernois [1992]). We further restrict our attention to the set of linear Markovian tax rules. That is, tax rules that are linear affine functions of the stock variable S: $\tau_i(S) = m_i + n_i\, S$, where m_i and n_i are constant parameters to be determined endogenously.

The timing of the game is as follows. At an initial stage, the social regulator sets and announces her/his output tax policy. Subsequently, the two duopolists compete in quantities over the infinite horizon of the game while taking the tax rule $\tau_i(S)$ as given. Formally, firm i maximizes the value of its stream of discounted profits subject to the law of evolution of the pollution stock:

$$\max_{q_i(.)\geq 0} \quad \Pi_i = \int_0^{+\infty} \left(\pi_i(t) - \tau_i(S)\, q_i\right)\, e^{-rt}\, dt. \tag{12}$$
$$s.t. \quad \dot{S}(t) = \left[s_g\, q_g(t) + s_b\, q_b(t)\right] - \delta S(t), \quad S(0) = \bar{S} \geq 0,$$

In solving (12) we assume that firm i uses feedback strategies; i.e., each firm condition its output decisions exclusively on the level of the pollution stock. Hence, the solution to problem (12) yields firm i's reaction function as $q_i(t) = q_i^*\,(S(t);\, m_i,\, n_i, m_j, n_j)$. The social planner chooses the tax parameters so as to maximize social welfare subject to the stock dynamics and taking the duopolists' output strategies as given:

$$\max_{(m_i,n_i,m_j,n_j)\in\Re} \quad W_l = \int_0^{+\infty} w_l(t)\, e^{-rt}\, dt. \tag{13}$$
$$s.t. \quad \dot{S}(t) = \left[s_g\, q_g(t) + s_b\, q_b(t)\right] - \delta S(t), \quad S(0) = \bar{S} \geq 0,$$
$$q_i(t) = q_i^*(S(t),\, m_i,\, n_i,\, m_j, n_j),\ \forall i (i \neq j) = b, g.$$

Solving the above problem yields the optimal Markovian tax (or subsidy) policy.

Simplifications In the remainder of this paper we restrict our attention to a reduced, and more analytically tractable, version of the above model which is obtained by setting[8]

$$s_g = 0,\, s_b = 1; \quad k_b = 0,\, k_g = k > 0; \quad \tilde{\theta} = 0,\, \hat{\theta} = 1. \tag{14}$$

[7] Indeed, we assume that $\tau_i(S)$ is not restricted in sign.

[8] It is important to note that the main ingredients of the general model are preserved under these assumptions: i) The heterogeneity among firms is retained as well as the possibility for the consumer to compare the two products along both horizontal and vertical dimensions of product differentiation. ii) The trade-off faced by the players between private and social costs is emphasized. The cost of producing the "brown" good and the emission/output ratio of the "green" firm are normalized to zero. In other words, the "green" good is costly to produce but involves no pollution emissions. By contrast, the brown good is produced at no (production) cost but entails two kinds of costs: i) a *private cost* corresponding to the reduced willingness to pay of the consumer for its product q_g and a *social cost* corresponding to the monetary value of the damage generated by the stock pollutant. iii) Consumers substitute away from the polluting good. Consumers are identified by their degree of environmental consciousness θ which is now uniformly distributed over the interval $[0, 1]$. The average consumer is now located at $\bar{\theta} = (1/2)$. Emission differentials reduce to $\Delta_g = -1$ and $\Delta_b = +1$. Consequently, the willingness to pay for the "green" ("brown") product increases (decreases) as the environmental problem becomes more severe. iv) Finally, since the "green" firm uses a 0-emission technology, its production does not add to the stock pollutant. Consequently, the law of motion of the pollution stock becomes $\dot{S}(t) = q_g(t) - \delta\, S(t)$. This simplifying assumption renders the asymmetric differential duopoly game analytically tractable and makes it possible to derive closed-form expressions of the asymmetric Markov perfect Nash equilibrium.

4 Paternalist regulation

In this section, we characterize the optimal markovian price policy assuming that the environmental regulator behaves as a paternalist regulator.

4.1 The social optimum

Before analyzing the environmental regulation game, it is useful to characterize the social optimum where firms can be directly controlled by the regulator. This solution provides a relevant benchmark against which the outcome of the environmental regulation game will be evaluated. It is obtained as the solution of the infinite-horizon control problem (13) in which $S(t)$ is the state variable and individual output levels $(q_g(t), q_b(t))$ are the control variables. In order to actually solve for the social optimum we use the maximum principle (see, Léonard and Long [1992]). The current-value Hamiltonian corresponding to the social planner's problem (13) is defined as

$$H_o(t) := \left[U_o(t) - k\, q_g(t) - (1/2)\, S(t)^2\right] + \lambda_o(t)\ [q_b(t) - \delta\, S(t)], \tag{15}$$

where $\lambda_o(t)$ denotes the co-state (or adjoint) variable associated with the pollution stock, $S(t)$. Assuming interior solutions, the Maximum principle implies that the following optimality conditions must hold[9]

$$\begin{aligned} k &= a - q_g(t) - \gamma\, q_b(t), && (16)\\ -\lambda_o(t) &= a - q_b(t) - \gamma\, q_g(t), && (17)\\ \dot{\lambda}_o(t) &= (r+\delta)\,\lambda_o(t) + S(t), && (18)\\ 0 &= \lim_{t\to+\infty} e^{-r\,t}\lambda_o(t)\, S(t) && (19) \end{aligned}$$

along with the dynamics of the stock pollutant (1).

The economic interpretation of the above conditions is simple. From conditions (16) and (17), firms are allocatively efficient when their prices are equal to the marginal social cost of production. Two remarks are in order, however. First, the marginal social cost of producing the green product boils down to the private marginal cost k whereas the marginal cost of producing the brown product corresponds to the shadow cost of the pollution stock, $-\lambda_o(t) > 0$. In other words, these two conditions highlight the trade-off faced by the paternalist regulator. She or he has to allocate production between a green firm – whose product is costly to manufacture and generates no pollution emissions– and a brown firm – whose product comes at a negligible cost but generates socially harmful pollution emissions that accumulate into the natural environment and causing present as well as long term environmental damages.

Second, the prices that appear on the right-hand side of conditions (16) and (17) are the '*correct market prices*'. In other words, the prices that would prevail should there be no adjustement of consumers' preferences after the optimal policy is implemented. Those prices should be contrasted with *prevailing market prices* – prices that consumers and firms actually face on the market – which are given by: $p_g(t) = a + S(t) - q_g(t) - \gamma\, q_b(t)$ and $p_b(t) = a - S(t) - q_b(t) - \gamma\, q_g(t)$. We shall return to this point later.

Now, we proceed by rewriting conditions (16-18) as a dynamical system in $\lambda_o(t)$ and $S(t)$. Solving the system of short-term conditions (16-17) for $q_g(t)$ and $q_b(t)$ yields

$$q_g(t) = \frac{((a-k) - \gamma\, a)}{(1-\gamma^2)} - \frac{\gamma\,\lambda_o(t)}{(1-\gamma^2)}, \quad q_b(t) = \frac{(a - \gamma\,(a-k))}{(1-\gamma^2)} + \frac{\lambda_o(t)}{(1-\gamma^2)}. \tag{20}$$

Remark 1. *Observe that $q_b(t) - q_g(t) = \frac{k+\lambda_o(t)}{(1-\gamma)}$. Hence, the brown firm produces more (resp., less) than the green firm whenever the private marginal social cost of producing the green product k is higher (resp., lower) than the shadow cost of the pollution stock $-\lambda_o(t)$. Furthermore, since $q_b(t) - q_g(t)$ is proportional to $\frac{1}{(1-\gamma)}$, the less differentiated the products are, the more sensitive the output differential is to the difference in effective marginal costs of production (here, $k + \lambda_o(t)$). This highlights how the trade-off between reduced environmental impact and increased production cost combines with the consumers' preference for variety (measured by the parameter of horizontal product differentiation γ) into the Paternalist regulator's decision problem.*

[9] In what follows, we shall omit the time argument when this does not create ambiguities.

Using Equations (20) to eliminate $q_g(t)$ and $q_j(t)$ from (1) and (18), we obtain a system of first-order linear differential equations

$$\begin{pmatrix} \dot{S}(t) \\ \dot{\lambda}_o(t) \end{pmatrix} = \begin{pmatrix} -\delta & \frac{1}{(1-\gamma^2)} \\ 1 & r+\delta \end{pmatrix} \cdot \begin{pmatrix} S(t) \\ \lambda_o(t) \end{pmatrix} + \begin{pmatrix} \frac{a-\gamma(a-k)}{(1-\gamma^2)} \\ 0 \end{pmatrix}, \tag{21}$$

which can be rewritten in compact form as $\dot{y}(t) = G_o\, y_o(t) + M_o$.

We are now in a position to assess the dynamic properties of the socially optimal solution. To start with, we characterize the steady-state solution. Observe that $|G_o| = -\delta\,(r+\delta) + 1/\left(\gamma^2 - 1\right) < 0$. Setting $\dot{y}(t) = 0$ into equation (21), and using the Cramer's rule to solve the resulting matrix equation $(G_o\, y_o(t) + M_o = 0)$ yields the steady-state values of the pollution stock and the associated shadow cost.

Proposition 1. *The steady-state values of the pollution stock and the associated shadow cost are given by*

$$S_o^\infty = \frac{(a - \gamma\,(a-k))\,(r+\delta)}{(\gamma^2 - 1)\,|G_o|} > 0, \quad \lambda_o^\infty = -\frac{S_o^\infty}{(r+\delta)} < 0. \tag{22}$$

Corresponding steady-state output levels are then easily derived. Setting the time derivative equal to zero in the dynamics of pollution accumulation yields $\hat{q}_b^\infty = \delta\, S_o^\infty$. Indeed, in the long-term, the stock of pollution will stabilize if the polluting firm emits no more than what the natural environment is able to absorb. Plugging $\hat{q}_b^\infty$ into condition (16) yields $\hat{q}_g^\infty = (a-k) - \gamma\,\delta\, S_o^\infty = (a-k) - \gamma\,\hat{q}_b^\infty$.

We now proceed with the characterization of the socially optimal time-path of production that ensures the convergence of $S(t)$ to S_o^∞. From the theory of differential equations, the solution to the system (21) endowed with the initial condition $S(0) = \bar{S} \geq 0$ is of the following form: $S_o(t) = (\bar{S} - S_o^\infty)e^{\rho t} + S_o^\infty$. The characteristic equation is defined as $|\rho\, I - G_o| = 0$, where

$$|\rho\, I - G_o| = \rho^2 - Tr(G_o)\,\rho + |G_o| = (\rho+\delta)\,(\rho - \delta - r) + 1/\left(\gamma^2 - 1\right). \tag{23}$$

Since $|G_o| < 0$, the characteristic equation admits two real roots of opposite sign, confirming a saddle point solution. The positive root corresponds to a diverging branch of the saddle point and is ruled out by the transversality condition. Hence, the unique trajectory that converges to the saddle point, for every initial stock of pollution $\bar{S}$, is defined by the negative root $\hat{\rho} = (1/2)\left(r - \sqrt{r^2 - 4\,|G_o|}\right)$. The following proposition summarizes our results:

Proposition 2. *There exists a unique (globally stable) social optimum. The socially optimal time-path of pollution accumulation is given by*

$$S_o(t) = (\bar{S} - S_o^\infty)e^{\hat{\rho} t} + S_o^\infty \tag{24}$$

and converges asymptotically to the steady-state value

$$S_o^\infty = \frac{(a - \gamma\,(a-k))\;(r+\delta)}{(1-\gamma^2)\;\delta\;(r+\delta) + 1} \quad \text{at a speed} \quad \hat{\rho} = \frac{1}{2}\left[r - \sqrt{(r+2\delta)^2 + 4/(1-\gamma^2)}\right]. \tag{25}$$

The corresponding time-paths of production are given by

$$\hat{q}_b(t) = \delta\, S_o^\infty + (\hat{\rho}+\delta)\,(\bar{S} - S_o^\infty)\, e^{\hat{\rho}\, t} \quad \text{and} \quad \hat{q}_g(t) = (a-k) - \gamma\,\hat{q}_b(t) \tag{26}$$

and converge to $\hat{q}_b^\infty = \delta S_o^\infty$ *and* $\hat{q}_g^\infty = (a-k) - \gamma\,\hat{q}_b^\infty$.

Note that $S_o^\infty > 0$. Since $\dot{S}_o(t) = \hat{\rho}\,(\bar{S} - S_o^\infty)\, e^{\hat{\rho} t}$ the stock of pollution converges to its steady-state from above if $(\bar{S} - S_o^\infty) > 0$ and from below if $(\bar{S} - S_o^\infty) < 0$. From (26) it is clear that the two outputs are strategic substitutes, i.e., when one firm increases its production, the optimal response of the competitor is to decrease its own production[10]. Using Equation (24), the feedback representation of socially optimal time-paths of production is

$$\hat{q}_b\,[S_o(t)] = -\hat{\rho}\, S_o^\infty + (\hat{\rho}+\delta)\; S_o(t), \tag{27}$$

$$\hat{q}_g\,[S_o(t)] = (a-k) - \gamma\,\hat{q}_b\,[S_o(t)] = ((a-k) + \gamma\,\hat{\rho} S_o^\infty) - \gamma\;(\hat{\rho}+\delta)\, S_o(t), \tag{28}$$

[10]The notion of strategic substitutability (or complementarity) was introduced byBulow et al. [1985].

Furthermore, from the characteristic equation, it comes that

$$(\hat{\rho}+\delta) = -1/\left[\left(\gamma^2-1\right)(\hat{\rho}-\delta-r)\right] < 0. \tag{29}$$

and differentiating (27)-(28) with respect to the stock of pollution leads to

$$\partial\hat{q}_b\left[S_o(t)\right]/\partial S_o(t) = (\hat{\rho}+\delta) < 0, \quad \partial\hat{q}_b\left[S_o(t)\right]/\partial S_o(t) = -\gamma\left(\hat{\rho}+\delta\right) > 0, \tag{30}$$

$$\partial\left(\hat{q}_b\left[S_o(t)\right]+\hat{q}_g\left[S_o(t)\right]\right)/\partial S_o(t) = (1-\gamma)\left(\hat{\rho}+\delta\right) < 0. \tag{31}$$

The above derivatives describe how market supply should evolve as the environmental problem becomes more severe. Since the build-up of pollution emissions leads to an increase in the shadow cost of the pollution stock, it upsets the balance between output levels and production costs described in remark 1. To maintain the balance, production should gradually be reallocated from the brown producer to the green producer as the stock of pollution increases. This trade-off is emphasized by considering the conditions which must be satisfied to ensure that both firms are active in a context of pollution accumulation (i.e., if $S_o^\infty > \bar{S}$). Note that $\hat{q}_b(0) = \delta\bar{S} + \hat{\rho}\left(\bar{S}-S_o^\infty\right) > 0$. Furthermore, since the brown firm is the only polluter in the model, it must be the case that $\hat{q}_b^\infty = \delta S_o^\infty > 0$. Now, observe that the green firm is active at time $t=0$ if, and only if, $(a-k)/\gamma > \hat{q}_g(0)$. For $\hat{q}_g(t)$ increases monotonically over time, this previous condition also ensures that $\hat{q}_g^\infty > 0$.

Finally, in our model where products are horizontally differentiated, the attainment of allocative efficiency requires that the adjustment in the production of the clean good less than compensate the contraction of the dirty product supply. More precisely, the reallocation process should aim at covering only a fraction γ of the reduction in production by the polluting firm. Hence, overall production decreases as pollution emissions accumulate into the ambient environment.

4.2 The optimal policy

We turn to the pollution control game and provide a characterization of the optimal environmental policy. We assume that the two duopolists use markovian strategies. Under this assumption, firm i assumes that its rival j conditions its strategy on the current level of the pollution stock; i.e., $q_j(t) := \Phi_j(S(t))$. Given the linear-quadratic structure of the game[11], we restrict our attention to linear markov strategies of the form $q_j(t) := \Phi_j(S(t)) = \phi_j + \psi_j\, S(t)$ where ϕ_j and ψ_j are constant coefficients. Similarly, the environmental regulator is assumed to rely on a linear markov tax (or subsidizations) scheme to regulate the market. In other words, the environmental regulator use firm specific taxes, and the tax charged to each firm i is a linear-affine function of pollution stock; i.e., $\tau_i(t) := T_i\left[S(t)\right] = m_i + n_i\, S(t)$ where m_i and n_i are unknown coefficients to be determined. Let $q_i^\star(t) = \phi_i^\star + \psi_i^\star S(t)$ denote the equilibrium strategy of firm i in the regulated duopoly game. By definition, the posited tax scheme decentralizes the social optimum as a Markov-Perfect Nash Equilibrium of the duopoly subgame if, and only if, $q_i^\star(t) = \hat{q}_i(t)$, $\forall i = b,\, g$ (or, equivalently, if $\phi_i^\star = \hat{\phi}_i$ and $\psi_i^\star = \hat{\psi}_i$, $\forall i = b,\, g$).

We are now in a position to characterize the unique tax scheme $\hat{\tau}_i(S(t))$ that decentralizes the social optimum. To begin with, let us consider the regulation of the green firm. Since the green firm does not pollute the environment and is at a cost disadvantage with respect to its competitor, we expect that the optimal tax policy will be negative implying a subsidy. In the duopoly subgame, the green firm simply maximizes $\Pi_g(t) = \left(a + (1/2)\,S(t) - q_g(t) - \gamma\, q_b(t)\right)\, q_g(t) - k\, q_g(t) - \tau_g(S(t))\, q_g(t)$, taking the current tax rate $\tau_i(S)$ and the output level of its competitor as given. Assuming interior solutions, the optimality condition reads as

$$a + (1/2)\,S(t) - 2\,q_g(t) - \gamma\, q_b(t) = k + \tau_g(S(t)). \tag{32}$$

This is the standard marginal revenue equals marginal cost condition for profit maximization. By definition, the optimal tax (or subsidy) rule $\tau^*(S(t))$ is such that the above condition matches the social optimality condition (16). Subtracting Equation (16) from Equation (32) yields $\hat{\tau}_g\left[S(t)\right] = -\hat{q}_g(t) + \frac{1}{2}\,S(t)$. Plugging the feedback representation of the socially optimal time-path of production into this expression and collecting with respect to $S(t)$ yields $\hat{\tau}_g[S(t)] = -\hat{\phi}_g + \left(\frac{1}{2} - \hat{\psi}_g\right) S_o(t)$. Finally, by identification, we get:

$$\hat{m}_g = -\hat{\phi}_g = -\left((a-k) + \hat{\rho}\,\gamma\, S_o^\infty\right), \quad \hat{n}_g = \frac{1}{2} - \hat{\psi}_g = \frac{1}{2} + \gamma\left(\hat{\rho}+\delta\right). \tag{33}$$

[11] Indeed,

We proceed by considering the regulation of the brown firm. Since the production of the brown product generates pollution emissions that accumulate over time, the brown firm faces an optimal control problem. The current value Hamiltonian associated with this problem is

$$H_b(t) = \left[\left(a - (1/2)\, S(t) - q_b(t) - \gamma\, q_g(t)\right)\, q_b(t) - \tau_b(S(t))\, q_b(t)\right] + \lambda_b\, [q_b(t) - \delta\, S(t)]\,. \tag{34}$$

Assuming interior solutions, the set of optimality conditions consists of the first-order conditions on production levels

$$\tau_b(S(t)) - \lambda_b(t) = -q_b(t) + \left(a - (1/2)\, S(t) - q_b(t) - \gamma\, q_g(t)\right), \tag{35}$$

$$\dot{\lambda}_b := r\,\lambda_b - \left(\tfrac{\partial H_b}{\partial S} + \tfrac{\partial H_b}{\partial q_g}\tfrac{\partial q_g}{\partial S}\right) = (r+\delta)\;\lambda_b(t) + \left(\frac{1}{2} + n_b - \gamma\,\psi_g\right)\, q_b(t), \tag{36}$$

$$0 = \lim_{t\to\infty} e^{-rt}\lambda_b\,(t)\, S\,(t) = 0 \tag{37}$$

together with the law of evolution of the pollution stock (1).

Again, the optimal tax rule $\hat{\tau}_g[S(t)]$ must ensure that the above conditions match the corresponding conditions for a social optimum (16-18). By identification of parameters, we obtain $\hat{m}_b$ and $\hat{n}_b$. The following proposition collects the results.

Proposition 3. *There exists a unique price-based policy that decentralizes the social optimum as a Markov-perfect Nash equilibrium of the duopoly game. The optimal pair of tax/subsidy rules is given by* $\hat{\tau}_i\,[S(t)] = \hat{m}_i + \hat{n}_i\, S(t)$, $i = b,\, g$ *where*

$$\hat{m}_g = -\left((a-k) + \hat{\rho}\,\gamma\, S_o^\infty\right) < 0, \qquad \hat{n}_g = \frac{1}{2} + \gamma\,(\hat{\rho}+\delta)\,, \tag{38}$$

$$\hat{m}_b = -\frac{\hat{\rho}\left((\rho - r) + \gamma^2\,(\rho+\delta)\right)\, S_o^\infty}{(r+\delta)} < 0, \qquad \hat{n}_b = -\frac{1}{2} + \frac{2}{(1-\gamma^2)\,(r+2\delta)} + \gamma^2\,(\hat{\rho}+\delta)\,. \tag{39}$$

Proof. See Appendix B. □

The signs of the above coefficients depend in a complex way on the values of the parameters of the model; i.e., a, k, γ, r, and δ. Note that $\hat{m}_b < 0$, and $\tilde{m}_g < 0$ (if we further assume that $q_g(0) = \Phi_g(0) > 0$; that is, the output level of the green firm is strictly positive when the stock of pollution is zero). For sufficiently low levels of pollution accumulation, the instantaneous rate of taxation is negative implying that both firms are subsidized. By contrast, the signs of $\hat{n}_b$ and $\hat{n}_g$ cannot be determined without additional assumptions. In the particular case where the goods are independent we have $\hat{n}_g|_{\gamma=0} = 1/2$. However, note that $\hat{n}_g'(\gamma) = (\hat{\rho}+\delta) + \gamma\hat{\rho}'(\gamma) < 0$ since

$$\hat{\rho}'(\gamma) = -\frac{2}{(1-\gamma)^3\sqrt{1/(8\gamma)^2 + (r+2\delta)^2}} < 0 \tag{40}$$

and $\lim_{\gamma\to 1^-}\hat{\rho}(\gamma) = -\infty$. In other words, $\hat{n}_g$ is decreasing in γ and becomes negative for sufficiently high levels of product differentiation. We conclude that whether the optimal tax rules are increasing or decreasing in the pollution stock cannot be determined *a priori*.

Finally, the possibility for the green firm to be taxed at the steady-state cannot be ruled out. Indeed, recall that $\tau_g\,(S(t)) = -q_g(t) + (1/2)S(t)$. Hence, we have $\tau_g\,(S_o^\infty) = -q_g^\infty + \frac{1}{2\delta}\, q_b^\infty = -(a-k) + \frac{(1+2\gamma\delta)}{2\delta} q_b^\infty$ and $\tau_g\,(S_o^\infty) \lesseqgtr 0$ if $q_b^\infty \lesseqgtr \frac{2\delta(a-k))}{(1+2\gamma\delta)}$.

5 Populist regulation

The purpose of this section is to characterize the tax/subsidy policy that would be chosen by a populist regulator. In order to simplify the exposition, we shall consider two scenarios that differ in the information available to the regulator. In the first one, we assume that the social planner observes the preferences of the average consumer only. Obviously, the preferences of the $\bar{\theta}$-type consumer can be represented by the following utility function:

$$U_a(t) = (a + S(t)/2)\; q_g(t) + (a - S(t)/2)\; q_b(t) - (1/2)\;\left(q_g^2(t) + q_b^2(t) + 2\gamma q_g(t) q_b(t)\right) + m. \tag{41}$$

Furthermore, we suppose that the social planner regards the $\bar{\theta}$-type consumer as a 'representative consumer'. Then, (instantaneous) social welfare is defined as $w_a(t) \equiv CS_a(t) + PS(t) - D(S(t))$ and can be rewritten as $w_a(t) = U_a(t) - k\, q_g(t) - (1/2)\, S(t)^2$. In the second scenario, we relax our assumption regarding the information that is available to the planner. Namely, we assume that the regulator has full knowledge of the distribution of consumers' preferences. Then, (instantaneous) social welfare[12] is given by $w_s(t) = U_a(t) - k\, q_g(t) - D(S(t)) - \Omega(S(t))$.

5.1 Restricted populist regulation

In this section, we characterize the optimal tax/subsidy scheme that decentralizes the restricted populist optimum as a Markov-perfect Nash equilibrium of the duopoly game. The restricted populist optimum is derived in Subsection (5.1.1). Following the same steps as in Subsection B, we characterize the optimal tax/subsidy scheme in Subsection (5.1.2).

5.1.1 The restricted social optimum

The current-value Hamiltonian corresponding to the social planner's problem (13) is defined as

$$H_a(t) := \left(U_s(t) - k\, q_g(t) - (1/2)\, S(t)^2\right) + \lambda_a(t)\ \left(q_b(t) - \delta\, S(t)\right), \tag{42}$$

where $\lambda_a(t)$ denotes the co-state variable associated with the pollution stock, $S(t)$. Assuming interior solutions, the maximum principle implies that the following first-order conditions must hold

$$k = a + (1/2)\, S(t) - q_g(t) - \gamma\, q_b(t), \tag{43}$$

$$-\lambda_a(t) = a - (1/2)\, S(t) - q_b(t) - \gamma\, q_g(t), \tag{44}$$

$$\dot{\lambda}_a(t) = (r+\delta)\, \lambda_a(t) - \frac{1}{2}\left(q_g(t) - q_b(t)\right) + S(t), \tag{45}$$

$$0 = \lim_{t \to +\infty} e^{-r\,t} \lambda_a(t)\, S(t). \tag{46}$$

along with the dynamics of the stock pollutant (1). Again, conditions (43) and (44) are the standard price equals marginal cost conditions. However, these conditions differ substantially from those obtained in the case of a Paternalistic regulation. Indeed, the prices that appear on the right-hand side of conditions (43) and (44) are the prevailing market prices rather than the '*correct price*'; i.e., the prices that would prevail in the absence of green consumerism as in conditions (16) and (17).

Now, we proceed by rewriting conditions (43-45) as a dynamical system in $\lambda_o(t)$ and $S(t)$. Solving the short-term equilibrium conditions (43) and (44) yields:

$$\tilde{q}_g(t) = \frac{((a-k) - \gamma\, a)}{(1-\gamma^2)} - \frac{\gamma\, \lambda_a(t)}{(1-\gamma^2)} + \frac{S(t)}{2(1-\gamma)}, \quad \text{and} \quad \tilde{q}_b(t) = \frac{(a - \gamma(a-k))}{(1-\gamma^2)} + \frac{\lambda_a(t)}{(1-\gamma^2)} - \frac{S(t)}{2(1-\gamma)}.$$

Remark 2. *Let $\Delta\tilde{q} \equiv \tilde{q}_b(t) - \tilde{q}_g(t)$. Observe that $\Delta\tilde{q} = \frac{k+\lambda_a(t)-S(t)}{(1-\gamma)}$. In the Paternalistic scenario (see remark 1), the tax differential $\Delta\hat{q}$ was a function of k and λ_o only. Here, note that the output differential, $\Delta\tilde{q}$, depends also on the observed level of pollution accumulation $S(t)$. Furthermore, observe that $\Delta\hat{q}(t) - \Delta\tilde{q}(t) = \frac{(\lambda_o(t)-\lambda_a(t))+S(t)}{(1-\gamma)}$.*

Rewriting conditions (43-45) as a dynamical system in $\lambda_a(t)$ and $S(t)$ yields $\dot{y}(t) = G_a\, y(t) + M_a$ where

$$G_a = \begin{pmatrix} -\frac{1}{2(1-\gamma)} - \delta & \frac{1}{(1-\gamma^2)} \\ 1 - \frac{1}{2(1-\gamma)} & r + \frac{1}{2(1-\gamma)} + \delta \end{pmatrix} \text{ and } M_a = \begin{pmatrix} \frac{a-\gamma(a-k)}{(1-\gamma^2)} \\ \frac{k}{2(1-\gamma)} \end{pmatrix}. \tag{47}$$

We are now in a position to characterize the steady-state solution. Solving the matrix equation $(G_a\, y(t) + M_a = 0)$, we get

$$S_a^\infty = \frac{(a-\gamma(a-k))(r+\delta)}{(\gamma^2-1)|G_a|} + \frac{(a-k)}{2(\gamma^2-1)|G_a|}, \quad \lambda_a^\infty = -\frac{k+(1-2\gamma)\, S_a^\infty}{(2(r+\delta)(1-\gamma)+1)}. \tag{48}$$

where $|G_a| = \frac{3+2(r+2\delta)(1+\gamma)}{4(\gamma^2-1)} - \delta\,(r+\delta) < 0$. Note that the steady-state pollution stock S_a^∞ is strictly positive. Setting the time derivative equal to zero in Equation (1) yields $\tilde{q}_b^\infty = \delta\, S_a^\infty$. Finally, plugging $\tilde{q}_b^\infty$ into condition (16) we get $\tilde{q}_g^\infty = (a-k) + (1/2 - \gamma\,\delta)\ S_a^\infty$.

[12] Note that $w_s(t) \equiv CS_s(t) + PS(t) - D(S(t)) = CS_o(t) + PS(t) - D(S(t)) - \Omega(S(t))$

We now turn to the characterization of the socially optimal time-path of production. The characteristic equation $|\rho\, I - G_a| = 0$ can be written as

$$(\rho + \delta)\,(\rho - \delta - r) + \tfrac{3+2(r+2\delta)(1+\gamma)}{4(\gamma^2-1)} \quad = \quad 0. \tag{49}$$

and admits two real roots of opposite sign, confirming a saddle point solution. The positive root corresponds to a trajectory that is ruled out by the transversality condition. Hence, the unique trajectory that converges to the saddle point for every initial stock of pollution $\bar{S}$ is the one that corresponds to the negative root $\tilde{\rho} = (1/2)\,\left(r - \sqrt{r^2 - 4\,|G_a|}\right)$.

Proposition 4. *There exists a unique (globally stable) social optimum. The socially optimal time-path of pollution accumulation $S_a(t) = (\bar{S} - S_a^\infty)e^{\tilde{\rho}\,t} + S_a^\infty$ converges asymptotically to the steady-state value*

$$S_a^\infty = \tfrac{2\,((a-k)+2\,(r+\delta)(a-\gamma(a-k)))}{3+2(1+\gamma)\delta(2(1-\gamma)(r+\delta)+r+2)} \quad \textit{at a speed} \quad \tilde{\rho} = \tfrac{1}{2}\left[r - \sqrt{(r+2\,\delta)^2 + \tfrac{3+2(r+2\delta)(1+\gamma)}{(1-\gamma^2)}}\right] \tag{50}$$

The corresponding time-paths of production are given by

$$\tilde{q}_b(t) = \delta\, S_a^\infty + (\tilde{\rho}+\delta)\,(\bar{S} - S_a^\infty)\, e^{\tilde{\rho}\,t} \textit{ and } \tilde{q}_g(t) = \Big((a-k) + \Big(\tfrac{1}{2} - \gamma\,\delta\Big)\; S_a^\infty\Big) + \Big(\tfrac{1}{2} - \gamma\,(\tilde{\rho}+\delta)\Big)\,(\bar{S} - S_a^\infty)\, e^{\tilde{\rho} t}. \tag{51}$$

The feedback representation of time-paths of production is given by:

$$\tilde{q}_b\,[S_a(t)] \quad = \quad -\tilde{\rho}\, S_a^\infty + (\tilde{\rho}+\delta)\; S_a(t), \tag{52}$$

$$\tilde{q}_g\,[S_a(t)] \quad = \quad ((a-k) + \gamma\,\tilde{\rho}\, S_a^\infty) + (1/2 - \gamma(\tilde{\rho}+\delta))\; S(t). \tag{53}$$

Aggregate output is given by:

$$\tilde{q}_b\,[S(t)] + \tilde{q}_c\,[S(t)] = ((a-k) - (1-\gamma)\,\tilde{\rho} S_a^\infty) + (1/2 + (\tilde{\rho}+\delta)\,(1-\gamma))\, S(t) \tag{54}$$

From the characteristic equation, it comes that

$$(\tilde{\rho} + \delta) = \frac{(3 + 2\,(r + 2\,\delta)\,(1+\gamma))}{4\,(1-\gamma^2)\,(\rho - \delta - r)} < 0. \tag{55}$$

and differentiating (52)-(53) with respect to the pollution stock leads to

$$\partial \tilde{q}_b[S(t)]/\partial S(t) \quad = \quad (\tilde{\rho}+\delta) < 0, \quad \partial \tilde{q}_g[S(t)]/\partial S(t)] = 1/2 - \gamma\,(\,\tilde{\rho}+\delta) > 0, \tag{56}$$

$$\partial\left(\tilde{q}_b\,[S(t)] + \tilde{q}_g\,[S(t)]\right)/\partial S(t) \quad = \quad 1/2 + (1-\gamma)\,(\tilde{\rho}+\delta)\,. \tag{57}$$

The same comments as in the previous section applies to the evolution of individual output levels. The above production rules indicate that the output level of the green firm should increase whereas that of the brown firm should decrease as pollution accumulates into the natural environment.

Remark 3. *We have $\partial\left(\tilde{q}_b\,[S(t)] + \tilde{q}_g\,[S(t)]\right)/\partial\,[S(t)] \lesseqgtr 0$ depending on whether $\gamma \lesseqgtr \frac{1}{2}$.*

Proof. Let $\partial\left(\tilde{q}_b\,[S(t)] + \tilde{q}_g\,[S(t)]\right)/\partial\,[S(t)] \equiv h(\gamma) = -\frac{3+2(1+\gamma)(\tilde{\rho}(\gamma)+\delta)}{4(1+\gamma)(r+\delta-\tilde{\rho}(\gamma))}$. We have $h(0) < 0$ and

$$h'(\gamma) = \tfrac{3(r+\delta-\tilde{\rho}(\gamma)) - (1+\gamma)(3+2(1+\gamma)(r+2\delta))\tilde{\rho}'(\gamma)}{4(1+\gamma)^2(r+\delta-\tilde{\rho}(\gamma))^2} > 0 \tag{58}$$

Hence, $h(\gamma)$ is monotonically increasing in γ and is bounded from below by $h(0) < 0$. Observe that $\tilde{\rho}(1/2) = -(1+\delta)$ so that $(\tilde{\rho}(1/2) + \delta) = -1$ and $h(1/2) = 0$. We conclude that $h(\gamma)$ is positive, negative or zero depending on whether $\gamma \gtreqless 1/2$. □

From the above remark, note that the aggregate output level may be increasing in the stock of pollution $S(t)$ if the degree of product differentiation is sufficiently high ($\gamma > \frac{1}{2}$). This result stands in sharp contrast with the previous case.

Remark 4. *We have $(\tilde{\rho} - \hat{\rho}) > 0$ and $|G_s| - |G_o| > 0$ if $(\gamma+1) < \frac{1}{2(r+\delta)}$. We have $(\tilde{\rho} - \hat{\rho}) < 0$ and $|G_s| - |G_o| < 0$ if $(\gamma+1) > \frac{1}{2(r+\delta)}$* 13

5.1.2 The restricted populist policy

To begin with, let us consider the regulation of the green firm. The profit maximization condition is given by Equation (32). The environmental regulator should set m_g and n_g in such a way that condition (32) matches the social optimality condition (43). Substracting Equation (32) from Equation (43) yields:

$$\tilde{\tau}_g(S) = -\tilde{q}_g(t). \tag{66}$$

It turns out that this suffices to conclude that the green firm is subsidized at any time $t \in [0, \infty)$. By identification, we obtain:

$$\tilde{m}_g = -\tilde{\phi}_g = -[(a-k) + \gamma\,\tilde{\rho}\,S_a^\infty] \quad \text{and} \quad \tilde{n}_g = -\tilde{\psi}_g = [\gamma\,(\tilde{\rho}+\delta) - \frac{1}{2}]. \tag{67}$$

Without loss of generality, we assume that $\Phi_g(0) > 0$; i.e., the output level of the green firm is strictly positive even though there is no pollution. This assumption suffices to ensure that $\tilde{m}_g$ is negative. Furthermore, note that $\tilde{n}_g < 0$ so that the green firm enjoys a subsidy whose rate increases as the stock of pollution increases.

We proceed by considering the regulation of the brown firm. The profit maximization conditions are given by Equations (36-36). Again, the optimal tax rule $\tau_g(S)$ must ensure that these conditions match the social optimality conditions (44-45). By identification of parameters, we obtain the following proposition:

Proposition 5. *There exists a unique price-based policy that decentralizes the social optimum as a Markov-perfect Nash equilibrium of the duopoly game. The optimal pair of tax/subsidy rules is given by* $\tau_i(S(t)) = \tilde{m}_i + \tilde{n}_i\,S(t)$, $i = b, g$ *where :*

$$\tilde{m}_g = -\left((a-k) + \tilde{\rho}\,\gamma\,S_s^\infty\right) < 0, \qquad \tilde{n}_g = \gamma\,(\tilde{\rho}+\delta) - \tfrac{1}{2} < 0, \tag{68}$$

$$\tilde{m}_b = -\tfrac{(a-k)}{2(r+\delta)} - \tfrac{\tilde{\rho}\left((\tilde{\rho}-r)+\gamma^2(\tilde{\rho}+\delta)\right)S^\infty}{(r+\delta)} < 0, \qquad \tilde{n}_b = \tfrac{(3/4-\delta(\tilde{\rho}+\delta))}{(r+2\delta)} + \tfrac{(\tilde{\rho}+\delta)\left((\tilde{\rho}-r)+\gamma^2(\tilde{\rho}+\delta)\right)}{(r+2\delta)} > 0. \tag{69}$$

Proof. See Appendix B.1 □

We have just shown that the restricted populist policy requires to subsidize the green firm over the whole horizon of the game. Turning to the regulation of the brown firm, note that $\tilde{m}_b$ is negative. Thus, the brown firm benefits from a subsidy provided that the stock of pollution is low enough. Since $\tilde{n}_b > 0$, this subsidy decreases over time as the stock of pollution increases.

5.2 Unrestricted regulation

In this section, we relax the restriction that the populist regulator ignores the distribution and support of consumers' preferences.

Proof. Observe that

$$(\tilde{\rho}-\hat{\rho}) > 0 \Leftrightarrow -\sqrt{r^2 - 4\,|G_s|} + -\sqrt{r^2 - 4\,|G_o|}, \tag{59}$$

$$\Leftrightarrow r^2 - 4\,|G_o| > r^2 - 4\,|G_s|\,, \tag{60}$$

$$\Leftrightarrow |G_s| > |G_o|\,, \tag{61}$$

and

$$|G_s| - |G_o| = \frac{(1 - 2\,(r+\delta)\,(1+\gamma))}{4\,(1-\gamma^2)}. \tag{62}$$

Note that $|G_s| - |G_o| > 0$ if, and only if, $(\gamma+1) < \frac{1}{2(r+\delta)}$. We conclude that $(\tilde{\rho}-\hat{\rho}) > 0$, if and only if $(\gamma+1) < \frac{1}{2(r+\delta)}$; i.e., if both the discount rate and the rate of pollution accumulation are small. □

Finally, observe that :

$$(S_s^\infty - S_o^\infty) = \frac{2\,(a-\gamma\,(a-k))\,(r+\delta)\,(|G_o| - |G_s|) + (a-k)\,|G_o|}{2\,(\gamma^2-1)\,|G_s|\,|G_o|}. \tag{63}$$

Therefore, we have

$$(S_s^\infty - S_o^\infty) > 0 \Leftrightarrow (|G_o| - |G_s|) < 0 \,\text{or}\, 0 > \frac{(|G_o| - |G_s|)}{|G_o|} > -\frac{(a-k)}{2\,(a-\gamma\,(a-k))\,(r+\delta)}, \tag{64}$$

$$(S_s^\infty - S_o^\infty) < 0 \Leftrightarrow 0 > -\frac{(a-k)}{2\,(a-\gamma\,(a-k))\,(r+\delta)} > \frac{(|G_o| - |G_s|)}{|G_o|}. \tag{65}$$

5.2.1 The populist social optimum

The current-value Hamiltonian corresponding to the social planner's problem (13) is defined as

$$H_p(t) := \left[U_p(t) - k\,q_g(t) - \tfrac{1}{2}\left(1 - \tfrac{1}{6(1-\gamma)}\right) S(t)^2\right] + \lambda_p(t)\ \left(q_b(t) - \delta\, S(t)\right), \tag{70}$$

where $\lambda_p(t)$ denotes the co-state variable associated with the pollution stock, $S(t)$. Again, we apply the Maximum Principle. The short-term optimality conditions are obtained by substitution of $\lambda_p(t)$ for $\lambda_a(t)$ in Equations (43) and (44). The adjoint equation

$$\dot{\lambda}_p(t) = (r+\delta)\,\lambda_p(t) - \tfrac{1}{2}\left(q_g(t) - q_b(t)\right) + S(t) - \tfrac{1}{6(1-\gamma)}\,S(t) \tag{71}$$

together with the transversality condition, $\lim_{t\to+\infty} e^{-r\,t}\lambda_a(t)\,S(t)$, and the law of evolution of the pollution stock complete the set of optimality conditions. At this point, one important remark may be made. If the populist regulator observes the distribution of consumers' preferences and use this observation to inform social decision-making, the rate of convergence and the steady-state of the dynamic optimisation problem obviously change. (?) However, it does not affect the expression of trajectories that we derived in the previous section. In particular, Remark 2 still holds. Namely, we have $\Delta q_p = q_b(t) - q_g(t) = \left(k + \lambda_p(t) - S(t)\right)/(1-\gamma)$ and $\Delta q^{obj} - \Delta q^{subj} = S(t)/(1-\gamma)$. Rewriting conditions (43-46) as a dynamical system in $\lambda_p(t)$ and $S(t)$ yields the matrix equation $\dot{y}(t) = G_p\,y(t) + M_p$ where

$$G_p = \begin{pmatrix} -\frac{1}{2(1-\gamma)} - \delta & \frac{1}{(1-\gamma^2)} \\ 1 - \frac{2}{3(1-\gamma)} & r + \frac{1}{2(1-\gamma)} + \delta \end{pmatrix} \quad \text{and} \quad m_p = \begin{pmatrix} \frac{a-\gamma(a-k)}{(1-\gamma^2)} \\ \frac{k}{2(1-\gamma)} \end{pmatrix}. \tag{72}$$

Solving the matrix equation $(G_p\,y(t) + M_p = 0)$ yields the steady-state values of the pollution stock and the associated shadow cost:

$$S_p^\infty = \frac{2\,a\,(r+\delta) + (a-k)\,(1 - 2\,\gamma\,(r+\delta))}{(\gamma^2-1)\left|G_p\right|}, \quad \lambda_p^\infty = \frac{k + (2(1-\gamma) + 2/3)S_p^\infty}{2(1-\gamma)(r+\delta) - 1}, \tag{73}$$

where $\left|G_p\right| = A - \delta\,(r+\delta)$ with $A = \frac{-7+9\gamma+6\,(\gamma^2-1)(r+2\delta)}{12\,(\gamma-1)^2(1+\gamma)}$.

The characteristic polynomial writes as $|\rho I - G_p| = (\check{\rho}+\delta)\,(\check{\rho} - \delta - r) + A$. The roots of the characteristic equation are then given by $\check{\rho}^{1,2} = \frac{1}{2}\left(r \pm \sqrt{(r+2\delta)^2 - 4\,A}\right)$. We restrict our attention to real roots; i.e, we assume that $(r+2\delta)^2 > 4\,A$. We have to consider two subcases. To begin with, let us assume that $|G_p| > 0$. Given that $Tr(G_p) = r > 0$ both roots are positive implying an unstable node. The corresponding trajectories are ruled out by the transversality condition. Now, let us assume that $|G_p| < 0$. In this case, the roots are of opposite sign, confirming a saddle point. Again, the positive root is ruled out by the transversality conditions and the stable branch of the dynamical system is obtained by choosing the negative root:

$$\check{\rho} = \frac{1}{2}\left(r - \sqrt{(r+2\delta)^2 - 4\,A}\right). \tag{74}$$

The following proposition summarizes our results.

Proposition 6. *Assume that $|G_p| < 0$. There exists a unique (globally stable) social optimum. The socially optimal time-path of pollution accumulation $S_p(t) = (\bar{S} - S_p^\infty)e^{\check{\rho}t} + S_p^\infty$ converges asymptotically to the steady-state value $S_p^\infty = \frac{2\,a\,(r+\delta)+(a-k)\,(1-2\,\gamma\,(r+\delta))}{(\gamma^2-1)(A-(r-1/(1-\gamma))\,\delta-\delta^2)}$ at a speed $\check{\rho} = \frac{1}{2}\left[r - \sqrt{(r+2\delta)^2 + \frac{7-9\gamma+6(r-2\delta)(1-\gamma^2)}{3(\gamma-1)^2(\gamma+1)}}\right]$. The corresponding time-paths of production are given by $\check{q}_b(t) = \delta\,S_p^\infty + (\check{\rho}+\delta)\left(\bar{S} - S_p^\infty\right)e^{\check{\rho}\,t}$ and $\check{q}_g(t) = (a-k) + \frac{1}{2}S(t) - \gamma\,\check{q}_b(t)$.*

The feedback representation of time-paths of production is given by:

$$\check{q}_b\left[S_p(t)\right] \;=\; -\check{\rho}\,S_p^\infty + (\check{\rho}+\delta)\ S_p(t), \quad \check{q}_g\left[S_p(t)\right] = \left((a-k) + \gamma\,\check{\rho}\,S_p^\infty\right) + \left((1/2) - \gamma\,(\check{\rho}+\delta)\right)\,S(t).$$

Aggregate output is given by:

$$\check{q}_b\left[S(t)\right] + \check{q}_c\left[S(t)\right] = \left((a-k) - (1-\gamma)\,\check{\rho}S_a^\infty\right) + \left(1/2 + (\check{\rho}+\delta)\,(1-\gamma)\right)S(t)$$

Note that $(\check{\rho} + \delta) \lesseqgtr 0$ depending on whether $A \lesseqgtr 0$. Thus, differentiating the quantities with respect to the pollution stock, we obtain

$$\begin{aligned} \partial \check{q}_b[S(t)]/\partial S(t) &= \check{\rho} + \delta < 0, \quad \partial \check{q}[S(t)]/\partial S(t)] = -\gamma\,(\check{\rho} + \delta) > 0, &(75)\\ \partial\left(\check{q}_b[S(t)] + \check{q}_b[(S(t)]\right)/\partial S(t) &= (1-\gamma)\,(\check{\rho} + \delta) < 0. &(76)\end{aligned}$$

5.2.2 The populist policy

From the previous subsection, recall that the diversity of consumers' preferences does not affect the optimality conditions for the green firm. Consequently, the optimal tax rule satisfies:

$$\check{\tau}_g(S) = -\check{q}_g(t). \tag{77}$$

It turns out that this suffices to conclude that the green firm is subsidized at any time $t \in [0, \infty)$. By identification, we obtain:

$$\check{m}_g = -\check{\phi}_g = -[(a-k) + \gamma\,\check{\rho}\,S_p^{\infty}] \quad \text{and} \quad \check{n}_g = -\check{\psi}_g = [\gamma\,(\check{\rho} + \delta) - 1/2]. \tag{78}$$

Without loss of generality, we assume that $\Phi_g(0) > 0$; i.e., the output level of the green firm is strictly positive even though there is no pollution. This assumption suffices to ensure that $\tilde{m}_g$ is negative. Furthermore, note that $\tilde{n}_g < 0$ so that the green firm enjoys a subsidy whose rate increases as the stock of pollution increases.

We proceed by considering the regulation of the brown firm. The profit maximization conditions are still given by Equations (35-37). Again, the optimal tax rule $\tau_g(S)$ must ensure that these conditions match the social optimality conditions (43-44) and (71). By identification of parameters, we obtain the following proposition:

Proposition 7. *There exists a unique price-based policy that decentralizes the social optimum as a Markov-perfect Nash equilibrium of the duopoly game. The optimal pair of tax/subsidy rules is given by* $\tau_i(S(t)) = \check{m}_i + \check{n}_i\,S(t)$, $i = b, g$, *where :*

$$\check{m}_g = -\left((a-k) + \check{\rho}\,\gamma\,S_s^{\infty}\right) < 0, \qquad \check{n}_g = \gamma\,(\check{\rho} + \delta) - \tfrac{1}{2} < 0, \tag{79}$$

$$\check{m}_b = -\frac{(a-k)}{2(r+\delta)} - \frac{\check{\rho}\big((\check{\rho}-r)+\gamma^2(\check{\rho}+\delta)\big)S^{\infty}}{(r+\delta)} < 0, \qquad \check{n}_b = \frac{7-9\gamma-12\,(1-\gamma)\,\delta\,(\check{\rho}+\delta)}{12\,(1-\gamma)\,(r+2\,\delta)} + \frac{(\check{\rho}+\delta)\big((\check{\rho}-r)+\gamma^2(\check{\rho}+\delta)\big)}{(r+2\delta)} \tag{80}$$

Proof. See Appendix B.2 . □

A sufficient condition for having $\tilde{n}_b$ positive is that $\gamma < \frac{7}{9}$.

6 Optimal regulation when lifestyles change drastically

In the remainder of this section, we assume that the two goods are independent; i.e., we restrict our attention to the limit case where $\gamma = 0$. A possible interpretation for this scenario is as follows. The two goods are sold on different markets and are associated with different lifestyles. The brown good is sold on the consumerist segment of the market. This product displays an iconic brand name (such as 'Mercedes', 'Coca-Cola' or 'Apple' for example) or logo (such as the Izod crocodile or the nike 'Swoosh'). Consumers value the brown good not only because of its functional value but also for its perceived status-symbolism appeal. In other words, they buy the branded product because they want to show that they belong to a particular social group or to use the product for symbolic self-extension. By contrast, the green product is sold on the anti-consumerist segment of the market and carries a 'no-name' brand or a green sticker (such as the E.U. ecolabel). As such, the consumption of the green product is associated with an alternative lifestyle: 'simple living' or 'eco-conscious'. We assume that consumers' preferences exhibit a marked preference for variety so that they consume both types of goods. However, as the environmental problem becomes more severe, they show less identification with the commercial brand name and attribute less value to the perceived status-symbolism appeal of the brown good. By contrast, their valuation of the environmentally friendly product increases and they find increasing opportunities of self-extension in the alternative lifestyle movement.

To simplify the presentation of the results, the following propositions are expressed in terms of a subsidy policy. In other words, the sign of the economic instrument is the converse of the one used in the previous section.

6.1 Regulation by a paternalist regulator

To begin with let us characterize the social optimum under the assumption that $\gamma = 0$. The following corollary to Theorem 1 can be stated:

Corollary 1. *There exists a unique (globally stable) social optimum. The socially optimal time time-path of pollution accumulation,* $S_o(t) = (\bar{S} - S_o^{\infty})e^{\hat{\rho}t} + S_o^{\infty}$*, converges asymptotically to the steady-state value* $S_o^{\infty} = \frac{a\,(r+\delta)}{\delta\,(r+\delta)+1} > 0$ *at a speed* $\hat{\rho} = \frac{1}{2}\left[r - \sqrt{(r+2\delta)^2+4}\right] < 0$*. The corresponding time-paths of production are given by* $\hat{q}_b(t) = \delta\, S_o^{\infty} + (\hat{\rho}+\delta)\,(\bar{S} - S_o^{\infty})\, e^{\hat{\rho}\,t}$ *and* $\hat{q}_g(t) = (a-k)$*, and converge to* $\hat{q}_b^{\infty} = \delta S_o^{\infty}$ *and* $\hat{q}_g^{\infty} = (a-k)$*.*

Note that $\hat{\rho} < 0$ and $0 < S_o^{\infty} < a$. Furthermore, observe that $(\hat{\rho}+\delta) < 0$. The feedback representation of socially optimal time-paths of production is $\hat{q}_b(S) = -\hat{\rho}\, S_o^{\infty} + (\hat{\rho}+\delta)\; S_o(t)$ and $\hat{q}_g(S) = (a-k)$. Note that social efficiency requires a constant supply of the green product. By contrast, given that $\hat{q}_b'(S) = (\hat{\rho}+\delta) < 0$, the supply of the polluting good should decrease as the stock of pollution increases. The comparison of output levels is then straightforward. Three subcases must be distinguished. First, if $\hat{q}_g(0) = (a-k) \geq \hat{q}_b(0) = -\hat{\rho} S_o^{\infty}$ then $\hat{q}_g(S) \geq \hat{q}_b(S)$ for all $S \in [\bar{S},\, S^{\infty}]$. Second, if $(a-k) \leq \hat{q}_b^{\infty}$ then $\hat{q}_g(S) \leq \hat{q}_b(S)$ for all $S \in [\bar{S},\, S^{\infty}]$. Finally, if $\hat{q}_b^{\infty} < (a-k) < -\hat{\rho}\, S_o^{\infty}$, there exists a treshold S_1 such that $\hat{q}_g(S) \leq \hat{q}_b(S)$ for all $S \in [\bar{S},\, S_1]$ and $\hat{q}_g(S) > \hat{q}_b(S)$ for all $S \in (S_1,\, S^{\infty}]$. These three subcases are illustrated in Figure 1(a).

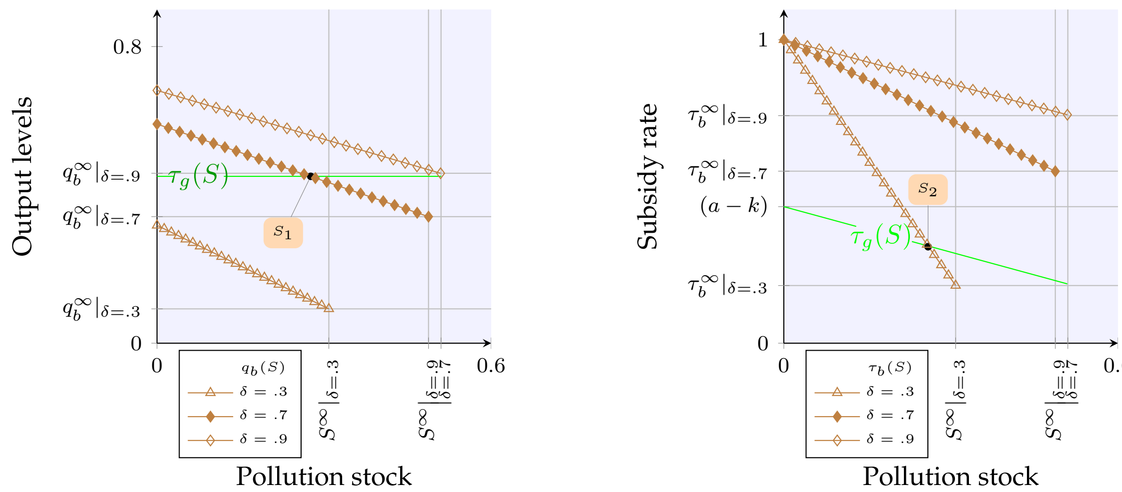

Figure 1: markovian production and subsidization rules

Turning to the prices, we have $\hat{p}_b\,(S) = (a + \hat{\rho}\, S_o^{\infty}) - \left(\hat{\rho}+\delta+\frac{1}{2}\right) S$ and $\hat{p}_g(S) = k + \frac{1}{2}\, S$. Note that $\hat{p}_b(0) = a + \hat{\rho}\, S_o^{\infty} > 0$ and $\hat{p}_g(0) = k{>}0$. Furthermore, observe that $\left(\hat{\rho}+\delta+\frac{1}{2}\right) \lesseqgtr 0$ depending on whether $(r+2\delta) \lesseqgtr \frac{3}{2}$. We conclude that the brown product price may be an increasing, constant or decreasing function of the stock of pollution depending on the parameter values. By contrast, the price of the green product increases with the stock of pollution. Also, note that $\delta < \frac{1}{4}$ implies $(r+2\,\delta) < \frac{3}{2}$ and $\left(\hat{\rho}+\delta+\frac{1}{2}\right) < 0$ whereas $\delta > \frac{3}{4}$ implies $(r+2\,\delta) > \frac{3}{2}$ and $\left(\hat{\rho}+\delta+\frac{1}{2}\right) > 0$. Hence, a low (resp., high) rate of regeneration of the natural environment is sufficient to ensure that the price of the brown product is decreasing (resp., increasing) in the stock of pollution. Finally, note that steady-state price levels are $\hat{p}_b^{\infty} = a - \left(\delta+\frac{1}{2}\right) S_o^{\infty} = \frac{a(2-(r+\delta))}{2(1+\delta(r+\delta))} \geq 0$ and $\hat{p}_g^{\infty} = k + \frac{1}{2} S_o^{\infty} > 0$. This last observation implies that the price of both products remains positive over time.

Let us now proceed with the analysis of the optimal policy scheme. Assuming that $\gamma = 0$, the following corollary to Theorem 2 obtains:

Corollary 2. *There exists a unique price-based policy that decentralizes the social optimum as a Markov-perfect Nash-equilibrium of the duopoly game. The optimal pair of subsidization rules is given by* $\tau_g\,[S(t)] = (a-k) - (1/2)\, S(t)$ *and* $\tau_b\,[S(t)] = a - \left[\frac{4-(r+2\delta)}{2(r+2\delta)}\right] S(t)$*. These rules converge to* $\tau_g^{\infty} = (a-k) - (1/2)\, S_o^{\infty}$ *and* $\tau_b^{\infty} = a - \left[\frac{4-(r+2\delta)}{2(r+2\delta)}\right] S_o^{\infty}$*.*

To begin with, note that $\tau_b(0) = a > \tau_g(0) = (a-k) > 0$. Since $\frac{4-(r+2\delta)}{2(r+2\delta)} > 0$, the optimal policy requires that both firms be granted a subsidy that decreases as the stock of pollution increases. If the initial stock of pollution $\bar{S}$ is sufficiently low, the brown firm initially benefits from a higher rate of subsidization. In order to determine whether the brown firm keeps this advantage over time, we have to compare the slopes of the two subsidization rules. Note that $\frac{4-(r+2\delta)}{2(r+2\delta)} \lesseqgtr \frac{1}{2}$ depending on whether $(r+2\delta) \gtreqless 2$. Hence, there are three possible cases depending on the parameters values. First, if $(r+2\delta) \geq 2$, the brown firm will keep this fiscal advantage over the whole horizon of the game. Second, if $(r+2\delta) < 2$, whether or not the fiscal advantage of the brown firm remains over time depends on the location of the intersection point between the two subsidization rules. Let us denote by S_2 the value of the pollution stock for which the identity $\tau_g(S) = \tau_b(S)$ holds. It is easy to see that $S_2 = \frac{k(r+2\delta)}{2(r+2\delta)}$. If $(r+2\delta) < 2$ and $S_2 > S_o^\infty$, the brown firm again keeps its advantage over the whole horizon of the game. However, if $(r+2\delta) < 2$ and if $S_2 < S_o^\infty$ the advantage reverses when the value of the pollution stock exceeds S_2; i.e., for all $S \in [\bar{S}, S_2]$ we have $\tau_b(S) \geq \tau_g(S)$ whereas for all $S \in (S_2, S_o^\infty]$ we have $\tau_b(S) < \tau_g(S)$.

At this point, it is important to note that $(r+2\delta) < 2$ for all $\delta \leq \frac{1}{2}$. Therefore, a low rate of pollution assimilation is sufficient to ensure that the brown firm will keep its fiscal advantage over time.

As is familiar in dynamic regulation problems, the optimal economic instrument may change in sign as the state-variable evolves over time[14]. As was shown above, the optimal instrument initially takes the form of a subsidy if $\bar{S}$ is sufficiently small. However, in the long-run, the optimal subsidization scheme may require that firms be taxed. We now investigate under which condition this case happens. To begin with, note that $\tau_b^\infty < 0$ if, and only if,

$$1 - \frac{(-r-2\delta+4)(r+\delta)}{2(r+2\delta)(\delta(r+\delta)+1)} < 0.$$

The values of r and δ for which this inequality holds are plotted on figure 2(a)[15]. A necessary condition for the above inequality to hold is $\delta < \delta_1 \simeq 0.2339$. Hence, we conclude that the brown firm may be taxed if, and only if, the environment' self-cleaning capacity is relatively small. Let us now turn to the green firm. Note that $\tau_g^\infty < 0$ if, and only if,

$$a < \frac{2k\,(1+\delta\,(r+\delta))}{2-(r+\delta)\,(1-2\delta)}.$$

Without loss of generality, let us normalize a to 1 so that $k \in [0,1)$ and the above inequality becomes:

$$1 - \frac{2k\,(1+\delta\,(r+\delta))}{2-(r+\delta)\,(1-2\delta)} < 0.$$

In figure 2(b), the above inequality is plotted in the (δ, r) plane for different values of k. Let us assume that $k = 11/20$ then the set of parameters for which the inequality holds is given by

$$\Omega|_{k=\frac{11}{20}} = \left\{(\delta,\, r)\,|\, 0 < \delta < \frac{1}{18}\left(1+\sqrt{37}\right),\, \frac{9}{10-9\delta} - \delta < r < 1\right\}.$$

[14]See, for example, Benchekroun and Long (1998) and Claude et al. (2010)

[15]Let us denote:

$$\delta_1 = \frac{1}{6}\left(\left(53-6\sqrt{78}\right)^{\frac{1}{3}} + \left(53-6\sqrt{78}\right)^{\frac{1}{3}} - 4\right),$$

$$\delta_2 = \frac{1}{4}\left(-1+\sqrt{49+32\sqrt{2}}\right),$$

$$r_1 = \frac{2-3\delta-6\delta^2}{2\,(1+2\delta)} - \frac{1}{2}\sqrt{\delta^2 - \frac{4\,(6\delta^2+3\delta-1)}{(1+2\delta^2)}},$$

$$r_2 = \frac{2-3\delta-6\delta^2}{2\,(1+2\delta)} + \frac{1}{2}\sqrt{\delta^2 - \frac{4\,(6\delta^2+3\delta-1)}{(1+2\delta^2)}}.$$

Then, it can be shown that the set of points for which $\tau_b^\infty < 0$ is given by :

$$\Theta = \{(r,\, \delta)|\, 0 \leq \delta < \delta_1,\, r_1 < r \leq 1 \vee \delta = \delta_1,\, r_1 < r < 1 \vee \delta_1 < \delta < \delta_2,\, r_1 < r < r_2\}$$

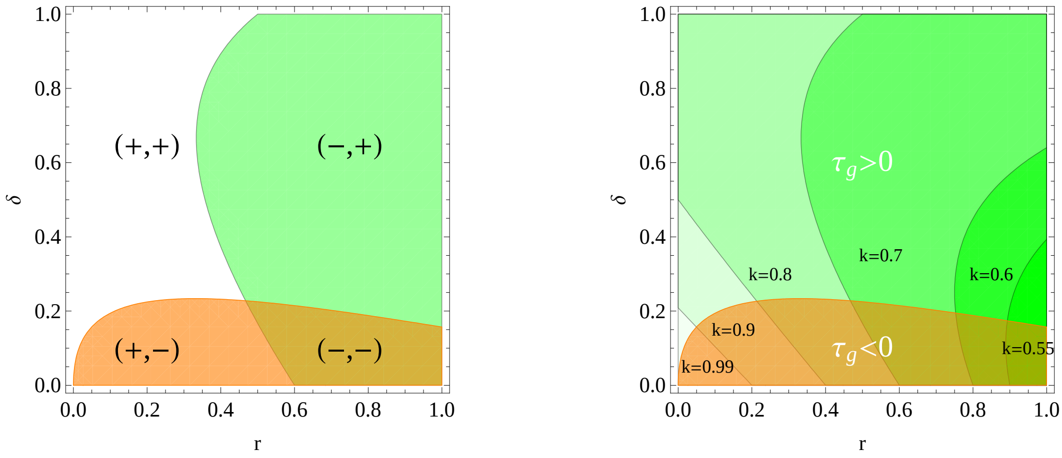

Figure 2: Sign of the economic intrument at the steady state

The boundary of this set is defined on figure 2(b) by the curve joining the points (0, 0.9) and (0.3935, 1). All points (r, δ) located to the right (left) of that curve are such that $\tau_g^\infty < (>) 0$. Now, let us assume that $k = 70/100$. Then, the set of parameters for which the above inequality holds becomes

$$\Omega_{k=\frac{70}{100}} = \left\{ (r,\, \delta)\,|\, 0 < \delta < 1,\, \frac{3}{5 - 3\delta} - \delta < r < 1 \right\}.$$

The boundary of $\Omega_{\frac{70}{100}}$ is defined on Figure 2(b) by the curve joining the points $(0, 3/5)$ and $(1, 1/2)$. Again, all points (r, δ) located to the right (left) of that curve are such that $\tau_g^\infty < (>) 0$. Note that $\Omega|_{k=\frac{11}{20}}$ is a subset of $\Omega_{\frac{70}{100}}$. It can be seen that the larger is k the larger is the set of parameters (r, δ) for which the inequality holds.

As is clear from figure 2 (a) no sign combination can be excluded at the steady state. Indeed, the optimal policy scheme may require to tax both firms at the steady state. Also, it may take the form of a carrott and stick, one firm being taxed while the other is subsidized. Obviously, in the latter case, the firm that should be taxed may be the brown firm. However, and more suprisingly, it may also be the green firm depending on the parameters values. At this point, it is important to note that the taxation of the green firm will occur if the green product is relatively expensive to produce (k is relatively large) and the rate of purification of the natural environment is high. By contrast, the taxation of the brown firm will occur if, and only if, the rate of purification of the natural environment is low.

6.2 Regulation by a populist regulator (unrestricted)

We follow the same steps as in the previous subsection. Assuming that $\gamma = 0$, the following corollary to Theorem 2 can be stated:

Corollary 3. *There exists a unique (globally stable) social optimum. The socially optimal time-path of pollution accumulation* $S_p(t) = S_p^\infty + (\bar{S} - S_p^\infty)e^{\check{\rho}t}$ *converges asymptotically to the steady-state value* $S_p^\infty = \frac{6[(a-k)+2a\,(r+\delta)]}{7+6\,(r+2\delta)+12\delta\,(r+\delta)} > 0$ *at a speed* $\check{\rho} = \frac{1}{2}\left[r - \sqrt{(r+2\delta)^2 + \frac{7+6(r+2\delta)}{3}}\right]$. *The corresponding time-paths of production are given by* $\check{q}_b(t) = \delta\, S_p^\infty + (\check{\rho} + \delta)(\bar{S} - S_p^\infty)e^{\check{\rho}\, t}$ *and* $\check{q}_g(t) = \left[(a-k) + (1/2)\, S_p^\infty\right] + (1/2)(\bar{S}\text{-}S_p^\infty)e^{\check{\rho}t}$.

Time-paths of production can be rewritten in feedback form as $\check{q}_b\left[S_p(t)\right] = -\check{\rho}\, S_s^\infty + (\check{\rho} + \delta)\, S_p(t)$ and $\check{q}_g\left[S_p(t)\right] = (a-k) + \frac{1}{2}\, S_p(t)$. Steady-state output levels are given by $\check{q}_b^\infty = \delta\, S_s^\infty$ and $\check{q}_g^\infty = (a-k) + (1/2)\, S_s^\infty$.

Remark 5. *We have* $\left(\check{\rho} + \delta + \frac{1}{2}\right) < 0$.

Proof. Straightforward. □

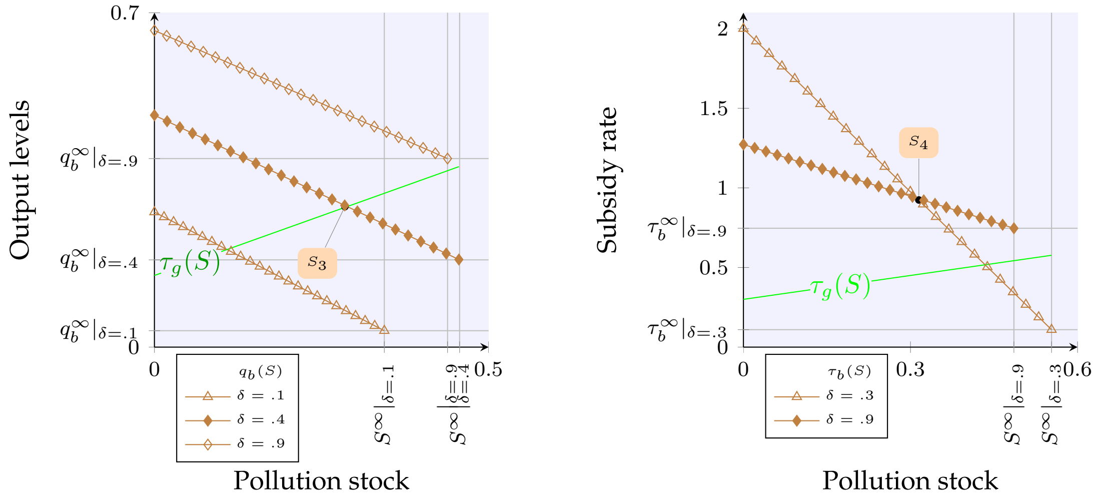

Figure 3: Markovian production and taxation rules

Obviously, the output level of the green (resp., brown) firm is increasing (resp., decreasing) in $S(t)$. For a given level of pollution accumulation $S(t)$, price levels are given by $\check{p}_b[S(t)] = \left(a + \check{\rho}\, S_p^\infty\right) - \left(\check{\rho} + \delta + \frac{1}{2}\right) S(t)$ and $\check{p}_g[S(t)] = k$. Along the socially optimal production path, the price of the green product remains constant and equal to the marginal production cost k. However, since $\left(\check{\rho} + \delta + \frac{1}{2}\right) < 0$, the price of the brown product decreases as the environmental problem becomes more severe. Steady-state price levels are :

$$\check{p}_b^\infty = a - \left(\delta + \frac{1}{2}\right) S_p^\infty = \frac{a\,(4a + 3k(r+\delta))}{7 + 6\,(r + 2\delta) + 12\delta\,(r+\delta)} \geq 0 \quad \text{and} \quad \check{p}_g^\infty = k > 0.$$

Let us now turn to the optimal subsidization policy. Given $\gamma = 0$, the following corollary to Theorem 3 can be stated:

Corollary 4. *There exists a unique price-based policy that decentralizes the social optimum as a Markov-perfect Nash equilibrium of the duopoly game. The optimal pair of tax/subsidy rules is given by* $\tau_g\,[S(t)] = (a-k) + (1/2)\,S(t)$ *and* $\tau_b\,[S(t)] = \left[a + \frac{(a-k)}{(r+\delta)}\right] - \left[\frac{1}{2} + \frac{7}{6(r+2\delta)}\right]\,S(t)$.

Observe that $\tau_b(0) = a + \frac{(a-k)}{(r+\delta)} > \tau_g(0) = (a-k)$. If the initial stock of pollution $\bar{S}$ is sufficiently low then both firms are initially granted a subsidy. Moreover, the brown firm initially benefits from a fiscal advantage over its competitor since $\tau_b(\bar{S}) > \tau_g(\bar{S})$. As pollution accumulates into the natural environment, the optimal subsidization scheme requires a decrease (resp., increase) in the rate of subsidization of the brown (resp., green) firm. Hence, the fiscal advantage may reverse as $S(t)$ increases. Let S_2 denote the level of pollution accumulation such that $\tau_b(S) = \tau_g(S)$. Straightforward computations yield $S_2 = \frac{6(r+2\delta)((a-k)+k(r+\delta))}{(r+\delta)(7+6(r+2\delta))}$. If $S_2 > S_p^\infty$ then the brown firm will benefit from a fiscal advantage over the whole horizon of the game. By contrast, if $S_p^\infty > S_2$, the fiscal advantage will turn to the green firm as soon as the stock of pollution exceeds the level S_2. Finally, note that the optimal policy scheme requires to subsidize the green firm over the whole horizon of the game whereas the brown firm may be taxed in the long-term. This will occur if the following inequality holds:

$$a + \frac{a-k}{r+\delta} - \frac{6(2(r+\delta)a + a - k)\left(\frac{1}{2} + \frac{7}{6(r+2\delta)}\right)}{12\delta(r+\delta) + 6(r+2\delta) + 7} < 0$$

The values of r and δ for which this inequality holds are plotted on figure 4.

In the long run, the brown firm will be taxed if the production of the green good is relatively unexpensive (k is small) and the rate of purification of the natural environment is relatively low (δ is small).

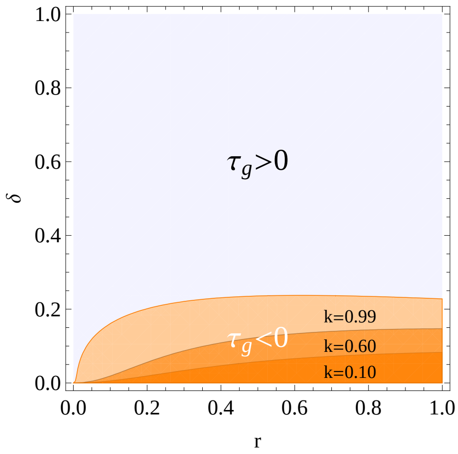

Figure 4: Sign of the economic instrument at the steady state

6.3 Regulation by a populist regulator (restricted)

Assuming that $\gamma = 0$, the following corrollaries are easily derived from theorems 4 and 5:

Corollary 5. *There exists a unique (globally stable) social optimum. The socially optimal time-path of pollution accumulation* $S_a(t) = (\bar{S} - S_a^\infty)e^{\tilde{\rho}\, t} + S_a^\infty$ *converges asymptotically to the steady-state value* $S_a^\infty = \frac{2[(a-k)+2\,a\,(r+\delta)]}{[3+2\,(r+2\,\delta)+4\,\delta\,(r+\delta)]}$ *at a speed* $\tilde{\rho} = (1/2)\,\left[r - \sqrt{(r+2\delta)^2 + 3 + 2(r+2\,\delta)}\right]$*. The corresponding time-paths of production are given by* $\tilde{q}_b(t) = \delta\, S_a^\infty + (\tilde{\rho} + \delta)\,(\bar{S} - S_a^\infty)\, e^{\tilde{\rho}\, t}$ *and* $\tilde{q}_g(t) = [(a-k) + (1/2)\; S_a^\infty] + (1/2)\, e^{\tilde{\rho} t}$*.*

It can be easily checked that $\check{\rho} > \tilde{\rho}$.

Corollary 6. *There exists a unique price-based policy that decentralizes the social optimum as a Markov-perfect Nash equilibrium of the duopoly game. The optimal pair of tax/subsidy rules is given by* $\tau_g\,[S(t)] = (a-k) + (1/2)\,S(t)$ *and* $\tau_b\,[S(t)] = \left[a + \frac{(a-k)}{(r+\delta)}\right] - \left[\frac{1}{2} + \frac{3}{2(r+2\delta)}\right]\; S(t)$*.*

The qualitative properties of the social optimum and of the optimal policy scheme are similar to those obtained in the previous scenario and do not require further comments. A few remarks are in order however. Let $\Delta_{p,a}^\infty = S_p^\infty - S_a^\infty$. Note that:

$$S_p^\infty - S_a^\infty = \frac{4\,[(a-k) + 2a\,(r+\delta)])}{(12\delta(\delta+1) + 6r(2\delta+1) + 7)(4\delta(\delta+1) + r(4\delta+2) + 3)} > 0.$$

Hence, a regulation policy based on the preferences of the average consumer only, leads to lower levels of pollution accumulation in the long-term. It follows that $\check{q}_b^\infty > \tilde{q}_b^\infty$, and using the fact that $\check{\rho} > \tilde{\rho}$, $\check{q}_b(0) > \tilde{q}_b(0)$. In other words, the output level of the brown firm in this scenario always exceeds the one obtained when the heterogeneity of consumers is taken into account. Also, we have $\check{q}_g(0) = \tilde{q}_g(0)$ and $\check{q}_g^\infty > \tilde{q}_g^\infty$. These observations jointly imply that $\check{p}_b(S) > \tilde{p}_b(S)$ for all admissible levels of the pollution stock.

Turning to the comparison of optimal subsidization schemes, note that $\check{\tau}_g(0) = (a-k) = \tilde{\tau}_g(0)$ and $\check{\tau}_g(S_a^\infty) > \tilde{\tau}_g(S_a^\infty)$. In other words, for all admissible level of the pollution stock $S(t) > 0$, the green firm benefits from a higher rate of subsidization when consumers' heterogeneity is ignored by the social planner. Turning to the brown firm, we have: $\check{\tau}_b(0) = a + \frac{(a-k)}{(r+2\delta)} = \tilde{\tau}_b(0)$ and $\check{\tau}_b^\infty \gtreqless \tilde{\tau}_b^\infty$ depending on whether $S_a^\infty \gtreqless (1/2)\,(3(r+2\delta) - 7)\,\Delta_{p,a}^\infty$.

6.4 Further comparisons

Figure 5 contrasts the paternalist and populist rules of taxation for identical parameters' values. This figure makes it clear that the two policy options may lead to opposite policy recommandations. For the chosen paramter values, in the long run, the paternalist policy provides higher subsidies to the green firm whereas the populist policy provides higher subsidies to the brown firm. Furthermore, under the paternalist policy, the green firm benefits from a higher rate of subsidization than the brown firm over the whole horizon of the game if the initial stock of pollution is sufficiently high $(\bar{S} > S_2)$. By contrast, under the populist policy, the brown firm will always benefit from higher subsidies.

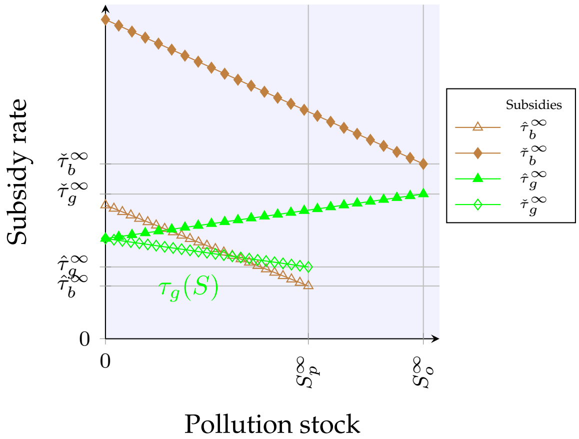

Figure 5: Comparison of paternalist and populist policy rules

In the remainder of this section, we compare steady state pollution levels. Let $\Delta S_{o,p}^{\infty} = S_o^{\infty} - S_p^{\infty}$. Note that

$$\frac{\partial \Delta S_{o,p}^{\infty}}{\delta k} = \frac{6}{7 + 6\left[(r + 2\delta) + 2\delta(r + \delta)\right]} > 0,$$

so that $\frac{\partial \Delta S_{o,p}^{\infty}}{\delta k}$ is increasing in k. Furthermore, note that

$$\Delta S_{o,p}^{\infty}|_{k=0} = \frac{a\left[6(r+\delta)^2 + 5(r+\delta) - 6\right]}{(1 + \delta(r+\delta))\ \left[7 + 6((r+2\delta) + 2\delta(r+\delta))\right]}$$

and that

$$k_1 = \frac{a\left[6(r+\delta)^2 + 5(r+\delta) - 6\right]}{6(1 + \delta(r+\delta))}.$$

Proposition 8. *Assume that $(r+\delta) \geq 3/2$ then $\Delta S_{o,p}^{\infty}$ is positive (or zero) for all $k \in [0, a[$. Assume that $(r+\delta) < 2/3$ and $r > 1/2$, then $\Delta S_{o,p}^{\infty}$ is negative for all $k \in [0, k_1[$ and positive for all $k \in [k_1, a[$.*

Proof. We have to consider three cases: i) if $\Delta S_{o,p}^{\infty}|_{k=0} \geq 0$ then $\Delta S_{o,p}^{\infty}$ is positive (or zero) for all $k \in [0,\ a[$; ii) if $\Delta S_{o,p}^{\infty}|_{k=0} < 0$ and $(a - k_1) > 0$ then $\Delta S_{o,p}^{\infty}$ is negative for all $k \in [0, k_1[$ and positive for all $k \in [k_1, a[$; iii) if $\Delta S_{o,p}^{\infty}|_{k=0} < 0$ and $(a - k_1) < 0$ then $\Delta S_{o,p}^{\infty}$ is negative for all $k \in [0, a[$.

Note that $\Delta S_{o,p}^{\infty}|_{k=0} \geq 0$ if $(r+\delta) \geq 3/2$. This is case 1. Given that $\Delta S_{o,p}^{\infty}$ is increasing in k, $\Delta S_{o,p}^{\infty}$ is positive (or zero) for all $k \in [0,\ a[$. Note that $\Delta S_{o,p}^{\infty}|_{k=0} < 0$ if $(r+\delta) < 3/2$. Furthermore, note that

$$(a - k_1) = \frac{a(r+\delta)(6(r+2\delta) - 5)}{6(\delta(r+\delta) + 1)}$$

is positive for $(r + 2\,\delta) > 5/6$. Hence, if $(r+\delta) < 3/2$ and $r > 1/2$, we have $\Delta S_{o,p}^{\infty}|_{k=0} < 0$ and $(a - k_1) > 0$. This is case2. Finally, since the conditions $(r+\delta) < 3/2$ and $(r + 2\,\delta) < 5/6$ are mutually incompatible, case 3 is impossible. □

Let $\Delta S_{o,a}^{\infty} = S_o^{\infty} - S_a^{\infty}$. Note that

$$\frac{\partial \Delta S_{o,a}^{\infty}}{\delta k} = \frac{2}{3 + 2\left[(r + 2\delta) + 2\delta(r + \delta)\right]} > 0,$$

so that $\frac{\partial \Delta S_{o,p}^{\infty}}{\delta k}$ is increasing in k. Furthermore, note that

$$\Delta S_{o,a}^{\infty}|_{k=0} = -\frac{a\left[2 + (r+\delta) - 2(r+\delta)^2\right]}{(1 + \delta(r+\delta))\ \left[3 + 2((r+2\delta) + 2\delta(r+\delta))\right]}$$

and that

$$k_2 = \frac{a\left[2 + (r+\delta) - 2(r+\delta)^2\right]}{2(1+\delta(r+\delta))}.$$

Proposition 9. *$\Delta S^{\infty}_{o,a}$ is positive (or zero) for all $k \in [0, a[$ if $(r+\delta) \geq (1/4) + \sqrt{17}/4$ and $r > -(3/4) + \sqrt{17}/4$. Assume that $(r+\delta) > 1/4 + \sqrt{17}/4$ and $r > -(3/2) + \sqrt{17}/4$, then $\Delta S^{\infty}_{o,a}$ is negative for all $k \in [0, k_2[$ and positive for all $k \in [k_2, a[$.*

Proof. We have to consider three cases: i) if $\Delta S^{\infty}_{o,a}|_{k=0} \geq 0$ then $\Delta S^{\infty}_{o,a}$ is positive (or zero) for all $k \in [0, a[$; ii) if $\Delta S^{\infty}_{o,a}|_{k=0} < 0$ and $(a - k_2) > 0$ then $\Delta S^{\infty}_{o,a}$ is negative for all $k \in [0, k_2[$ and positive for all $k \in [k_2, a[$; iii) if $\Delta S^{\infty}_{o,a}|_{k=0} < 0$ and $(a - k_2) < 0$ then $\Delta S^{\infty}_{o,a}$ is negative for all $k \in [0, a[$.

Note that $\Delta S^{\infty}_{o,a}|_{k=0} \geq 0$ if $(r+\delta) \geq (1/4) + \sqrt{17}/4$ and $r > -(3/4) + \sqrt{17}/4$. This is case 1. Given that $\Delta S^{\infty}_{o,a}$ is increasing in k, $\Delta S^{\infty}_{o,a}$ is positive (or zero) for all $k \in [0, a[$. Note that $\Delta S^{\infty}_{o,a}|_{k=0} < 0$ if $r < -(3/4) + \sqrt{17}/4$ or if $r > -(3/4) + \sqrt{17}/4$ and $(r+\delta) < (1/4) + \sqrt{17}/4$. Furthermore, note that

$$(a - k_2) = \frac{a(r+\delta)(2(r+2\delta) - 1)}{2(\delta(r+\delta)+1)}$$

is positive if $r \geq (1/2)$ or if $r < (1/2)$ and $(r + 2\delta) \geq 1/2$. Hence, if $(r+\delta) > 1/4 + \sqrt{17}/4$ and $r > -(3/2) + \sqrt{17}/4$, we have $\Delta S^{\infty}_{o,a}|_{k=0} < 0$ and $(a - k_2) > 0$. This is case 2. Finally, since the conditions are mutually incompatible, case 3 is impossible. □

7 Conclusion

We considered a dynamic model of pollution control in which the environmental regulator anticipates consumers' lack of responsiveness to environmental policy implementation. We characterized the optimal paternalist and populist environmental policies. It was shown that both policies require to subsidize the conventional firm and the green firm in the short-run. However, as is familiar in a dynamic setting, the optimal economic instrument may turn into a tax in the long-run for one (or both) firms. The change in the sign of the economic instrument was shown to depend on the cost involved in the production of the green product and the rate of purification of the natural environment. In a situation where the optimal (or paternalist) policy leads to a beneficial fiscal treatment for the green firm at the steady state, it was shown that the implementation of the populist policy may result in a higher rate of subsidization for the brown firm. Finally, under the populist policy, the accumulation of pollutants into the ambient environment may be excessive or insufficient depending on the parameters' values.

References

J. Alba and J.W. Hutchinson. Knowledge calibration: what consumers know and what they think they know. *Journal of consumer research*, 27:123–156, 2000.

Seema Arora and Shubhashis Gangopadhyay. Toward a theoretical model of voluntary overcompliance. *Journal of Economic Behavior & Organization*, 28(3):289–309, December 1995. ISSN 0167-2681.

Sangeeta Bansal and Shubhashis Gangopadhyay. Tax/subsidy policies in the presence of environmentally aware consumers. *Journal of Environmental Economics and Management*, 45(2, Supplement 1):333–355, March 2003. ISSN 0095-0696.

H. Benchekroun and N. Van Long. Efficiency inducing taxation for polluting oligopolists. *Journal of Public Economics*, 70(2):325–342, 1998.

R.J. Bord, R.E. O'Connor, and A. Fisher. In what sense does the public need to understand global climate change? *Public Understanding of Science*, 9:205–218, 2000.

Jeremy I. Bulow, John D. Geanakoplos, and Paul D. Klemperer. Multimarket oligopoly: Strategic substitutes and complements. *The Journal of Political Economy*, 93(3):488–511, June 1985. ISSN 00223808.

K. Conrad. Price competition and product differentiation when consumers care for the environment. *Environmental and Resource Economics*, 31:1–19, 2005.

H. Cremer and J.F. Thisse. On the taxation of polluting products in a differentiated industry. *European Economic Review*, 43:575–594, 1999.

A. Dixit. A model of duopoly suggesting a theory of entry barriers. *Bell Journal of Economics*, 10(1):20–32, Spring 1979. available at http://ideas.repec.org/a/rje/bellje/v10y1979ispringp20-32.html.

C. Eriksson. Can green consumerism replace environmental regulation ? a differentiated-product case. *Resource and Energy Economics*, 26:281–293, 2004.

Jonas Hackner. A note on price and quantity competition in differentiated oligopolies. *Journal of Economic Theory*, 93(2):233–239, August 2000. available at http://ideas.repec.org/a/eee/jetheo/v93y2000i2p233-239.html.

L. Karp and J. Livernois. On efficiency inducing taxation for a nonrenewable resource monopolist. *Journal of Public Economics*, 49(2):219–239, 1992.

C. Lombardini-Riipinen. Optimal tax policy under environmental quality competition. *Environmental and Resource Economics*, 32(3):317–336, November 2005.

D. Léonard and N. Van Long. *Optimal Control Theory and Static Optimization in Economics*. Cambridge University Press, Cambridge, United Kingdom, 1992.

P. Mahenc. Are green products over-priced ? *Environmental and Resource Economics*, 38:461–473, 2007.

Philippe Mahenc. Signaling the environmental performance of polluting products to green consumers. *International Journal of Industrial Organization*, 26(1):59–68, 2008. ISSN 0167-7187.

C. Manasakis, E. Mitrokostas, and E. Petrakis. Corporate social responsibility in oligopoly. Working Paper 07-07, University of Crete, Department of Economics, 2007.

Jose Luis Moraga-Gonzalez and Noemi Padron-Fumero. Environmental policy in a green market. *Environmental and Resource Economics*, 22(3):419–447, 2002.

M. Motta and J.F. Thisse. Minimum quality standards as an environmental policy: domestic and international effects. In E. Petrakis, E. Sartzetakis, and A. Xepapadeas, editors, *Environmental Regulation and Market Power*, Cheltenham, UK, 1999. E. Elgar.

E. Petrakis, E. Sartzetakis, and Xepapadeas. Environmental information provision as a public policy instrument. *Contributions to economic analysis and policy*, 4(1):Article 14, 2005a.

Emmanuel Petrakis, Eftichios Sophocles Sartzetakis, and Anastasios Xepapadeas. Environmental information provision as a public policy instrument. *The B.E. Journal of Economic Analysis & Policy*, 4(1), November 2005b.

P.R. Portney. Trouble in happyville. *Journal of Policy Analysis and Management*, 11:131–2, 1992.

F. Salanié and N. Treich. Regulation in happyville. *The Economic Journal*, 119:665–679, April 2009.

E.S. Sartzetakis, A. Xepapadeas, and E. Petrakis. The role of information provision as a policy instrument to supplement environmental taxes: Empowering consumers to choose optimally. Working Papers 307, Fondazione Eni Enrico Mattei, 2009.

N. Singh and X. Vives. Price and quantity competition in a differentiated oligopoly. *Rand Journal of Economics*, 15:546–554, 1984.

T.R. Walker, J.O. Habeck, T.P. Karjalainen, T. Virtanen, N. Solovenia, V. Jones, P. Kuhry, V.I. Ponomarev, K. Mikkola, A. Nikula, E. Patova, P.D. Crittenden, S.D. Young, and T. Ingold. Perceived and measured levels of environmental pollution: interdisciplinary research in the subartic lowlands of northeast european russia. *Ambio*, 35(5):220–228, August 2006.

A Consumers demand and consumers' surplus

A.1 Derivation of the inverse demand system

The θ-type consumer solves $\max_{x_g, x_b > 0} U(\theta, S)$ subject to $p_g\, x_g(\theta,\, S) + p_b\, x_b(\theta,\, S) + m = I$, where I is the consumer's income in units of the numeraire good m. The first-order conditions yield his/her inverse demand functions:

$$p_i(\theta,\, S) = a - \theta\, \Delta_i\, S - x_i(\theta,\, S) - \gamma\, x_j(\theta, S), \quad i(\neq j) = g,\, b. \tag{81}$$

Inverting the above system and using the identity $\Delta_j = -\Delta_i$, we obtain consumer θ's direct demand functions:

$$x_i(\theta, S) = \frac{a\,(1-\gamma) - \theta(1+\gamma)\,\Delta_i\, S}{(1-\gamma^2)} - \frac{p_i}{(1-\gamma^2)} + \frac{\gamma\, p_j}{(1-\gamma^2)}, \quad i(\neq j) = b, g. \tag{82}$$

Aggregate demand functions are obtained by summing individual demands. Recall that $(\hat{\theta} - \tilde{\theta}) = 1$. Also, note that $f(\theta) = 1/(\tilde{\theta} + 1 - \tilde{\theta}) = 1$ for all $\theta \in [\tilde{\theta}, \hat{\theta}]$ since consumers are uniformly distributed on this interval. Then, firm i's aggregate demand function is given by :

$$\begin{aligned} q_i(p_i,\, p_j,\, S) &\equiv \int_{\theta_1}^{\theta_2} x_i(\theta,\, S)\, f(\theta) d\theta = \int_{\theta_1}^{\theta_2} x_i(\theta,\, S)\, d\theta \\ &= \frac{a\,(1-\gamma) - \bar{\theta}(1+\gamma)\,\Delta_i\, S}{(1-\gamma^2)} - \frac{p_i}{(1-\gamma^2)} + \frac{\gamma\, p_j}{(1-\gamma^2)}, \quad i(\neq j) = b, g. \end{aligned} \tag{83}$$

where $\bar{\theta} = (\theta_1 + \theta_2)/2$ characterizes the average-type consumer. Finally, the inverse demand system is obtained by inverting (83):

$$p_i(q_i,\, q_j,\, S) = a - \bar{\theta}\, \Delta_i\, S - q_i - \gamma\, q_j, \quad i(\neq j) = b,\, g. \tag{84}$$

A.2 Derivation of the aggregate consumers' surplus

The net consumer surplus for a consumer of type θ is defined as $CS^s(\theta,\, t) \equiv U(\theta,\, t) - \left(p_g^s(t)\, x_g\,(\theta, t) + p_b^s(t)\, x_b\,(\theta,\, t)\right)$. By plugging (82) and (81) into the above expression, we get $CS^s(\theta, t) = \frac{1}{2}\left(x_g(\theta,\, t)^2 + x_b(\theta,\, t)^2 + 2\,\gamma\, x_g(\theta,\, t)\, x_b(\theta,\, t)\right)$. The aggregate consumers' surplus is then defined as the sum of individual consumer surpluses: $CS^s\,(t) \equiv \int_{\theta_1}^{\theta_2} CS^s(\theta, t)\, f(\theta)\, d\theta = \int_{\theta_1}^{\theta_2} CS^s(\theta, t)\, d\theta$. Substituting $x_g(\theta,\, t)$ for its value given in (82), integrating and using equation (84) to rearrange terms yield:

$$CS^s(t) = \frac{1}{2}\,(\theta_2 - \theta_1)\,\left(q_g\,(t)^2 + q_b\,(t)^2 + 2\,\gamma\, q_g\,(t)\; q_b\,(t)\right) + \frac{S(t)^2}{24\,(1-\gamma^2)}\left((\theta_2 - \theta_1)^3\left(\Delta_g^2 + \Delta_b^2 - 2\,\gamma\,\Delta_g\,\Delta_b\right)\right)$$

Finally, using $\Delta_g = -\Delta_b$ and $(\theta_2 - \theta_1) = 1$, the uninformed consumer surplus rewrites as

$$CS^s(t) = \frac{1}{2}\left(q_g\,(t)^2 + q_b\,(t)^2 + 2\,\gamma\, q_g\,(t)\; q_b\,(t)\right) + \left(\frac{\Delta_g^2\, S(t)^2}{12\,(1-\gamma^2)}\right).$$

We conclude that $CS^s(t) = CS^0(t) + \Omega(S(t))$ where $\Omega(S(t)) = \frac{\Delta_g^2}{12(1-\gamma^2)}\, S(t)^2$.

B Paternalist policy

In this appendix, we characterize the socially optimal policy scheme. The regulator chooses the taxation/subsidization rule $\hat{\tau}_b\,(S(t))$ so as to decentralize the social optimum. Formally, this amounts to choosing $\hat{\tau}_b\,(S(t))$ in such a way that the optimality conditions (35)-(37) match the conditions for a social optimum (16)-(19). Assuming that each firm uses Markovian strategies, recall that $q_i(t) \equiv \Phi_i\,(S(t))\,,\, i = b,\, g$. Using (17) to eliminate $\lambda_o(t)$ from Equation (18), we obtain the following conditon :

$$\left(\Phi_b'(S(t)) + \gamma\,\Phi_g'(S(t))\right)\dot{S}(t) = (r+\delta)\,\left(\Phi_b\,(S(t)) + \gamma\,\Phi_g\,(S(t)) - a\right) + S(t). \tag{85}$$

Similarly, using (35) to eliminate $\lambda_d(t)$ from Equation (36), we obtain:

$$\left(\tau_b'(S) + 1/2 + 2\,\Phi_b'(S) + \gamma\,\Phi_g'(S)\right)\dot{S} = (r+\delta)\,\left(\tau_b(S) - a + S/2 + 2\,\Phi_b(S) + \gamma\,\Phi_g(S)\right) + \left(1/2 + \gamma\,\Phi_g'(S) + \tau_b'(S)\right)\Phi_b(S) \quad (86)$$

Now, substracting Condition (86) from Condition (85), we get:

$$\left(\tau_b'(S) + \Phi_b'(S) + 1/2\right)\dot{S} - (r+\delta)\left(\tau_b(S) + S/2 + \Phi_b(S)\right) + S(t) - \left(1/2 + \gamma\,\Phi_g'(S) + \tau_b'(S)\right)\Phi_b(S) = 0. \quad (87)$$

Recall that socially optimal production rules are given by $\hat{\Phi}_i(S) = \hat{\phi}_i + \hat{\psi}_i\,S$, $i = b, g$. Furthermore, recall that the socially optimal time path of pollution accumulation is given by $S(t) = (\bar{S} - S_o^\infty)\,e^{\rho t} + S_o^\infty$ so that $\dot{S}(t) = \rho\,(\bar{S} - S_o^\infty)\,e^{\rho t}$. Assuming that $\tau_b\,(S(t)) = m_b + n_b\,S(t)$, Condition (87) can be rewritten as $\alpha_o(m_b, n_b) + \beta_o(m_b, n_b)\,(\bar{S} - S_o^\infty)e^{\rho t} = 0$ where

$$\alpha_o\,(m_b, n_b) = -2\,(r+\delta)\,m_b - \phi_b - 2\,\phi_b\,\left((r+\delta) + n_b + \gamma\,\psi_g\right) - \left((r+\delta) + \psi_b + 2\left((n_b + \psi_b)\,(r+\delta) + \psi_b(n_b + \gamma\psi_g) - 1\right)\right)$$

$$\beta_o(m_b, n_b) = -(r+\delta) + \rho - 2n_b\left((r+\delta) - \rho + \psi_b\right) - \psi_b\left(2\,(r+\delta) - 2\left(\rho - \gamma\psi_g\right) + 1\right) + 2$$

Note that this identity should hold at every time t. This will be the case if the tax parameters m_b and n_b are chosen so as to solve $\{\alpha_o(m_b, n_b) = 0,\ \beta_o(m_b, n_b) = 0\}$. We obtain :

$$\hat{m}_b = -\frac{\hat{\phi}_b\left((r+\delta)(r+\delta-\hat{\rho}) + \gamma\hat{\psi}_g(r+\delta-\hat{\rho}) + \hat{\rho}\hat{\psi}_b + 1\right)}{(r+\delta)\left((r+\delta-\hat{\rho}) + \hat{\psi}_b\right)} - \frac{\hat{\rho}\left(\hat{\psi}_b^2 - \gamma\hat{\psi}_g\hat{\psi}_b + 1\right)S_o^\infty}{(r+\delta)\left((r+\delta-\hat{\rho}) + \hat{\psi}_b\right)}$$

$$\hat{n}_b = -\frac{(1/2)(r+\delta-\hat{\rho}) + \hat{\psi}_b\left[\left((r+\delta-\hat{\rho}) + \gamma\hat{\psi}_g\right) + (1/2)\right] - 1}{\left((r+\delta-\hat{\rho}) + \hat{\psi}_b\right)}$$

Finally, Plugging $(\hat{\phi}_g, \hat{\psi}_g, \hat{\phi}_b, \hat{\psi}_b)$ back into the expressions for $\hat{m}_b$ and $\hat{n}_b$, and using the fact that $\rho^2 = r\rho + \delta(r+\delta) + 1/(\gamma^2 - 1)$ to rearrange terms, we obtain Equations (38-39).

B.1 Populist policy (restricted)

We follow the same steps as in the previous section. Using (44) to eliminate $\lambda_a(t)$ from Equation (46), we obtain the following conditon :

$$\dot{S}\,\left(1/2 + \Phi_b'(S) + \gamma\,\Phi_g'(S)\right) - (r+\delta)\left(-a + S/2 + \Phi_b(S) + \gamma\,\Phi_g(S)\right) + (1/2)\left(\Phi_g(S) - \Phi_b(S)\right) - S = 0$$

Now, substracting Condition (86) from Condition (85), we get:

$$\dot{S}\,\left(\tau_b'(S) + \Phi_b'(S)\right) - (r+\delta)\left(\tau_b(S) + \Phi_b(S)\right) - \left(\gamma\,\Phi_g'(S) + \tau_b'(S)\right)\Phi_b(S) - (1/2)\,\Phi_g(S) + S = 0$$

Condition (87) can be rewritten as $\alpha_a(m_b, n_b) + \beta_a(m_b, n_b)\,(\bar{S} - S_o^\infty)e^{\rho t} = 0$ where

$$\alpha_a(m_b, n_b) = -2(r+\delta)m_b - \phi_g - 2\phi_b\left(r+\delta+n_b+\gamma\psi_g\right) - S_\infty\left(2n_b\left(r+\delta+\psi_b\right) + \psi_g + 2\psi_b\left(r+\delta+\gamma\psi_g\right) - 2\right),$$
$$\beta_a(m_b, n_b) = -2n_b\left(r+\delta-\rho+\psi_b\right) - \psi_g - 2\psi_b\left(r+\delta-\rho+\gamma\psi_g\right) + 2.$$

Solve $\{\alpha_o(m_b, n_b) = 0,\ \beta_o(m_b, n_b) = 0\}$ for $\tilde{m}_b$ and $\tilde{n}_b$. We obtain :

$$\tilde{m}_b(\phi_b, \psi_b, \phi_g, \psi_g) = \frac{\phi_b\left(-2(r+\delta)(r+\delta-\rho) - 2\rho\psi_b + (1 - 2\gamma(r+\delta-\rho))\psi_g - 2\right) - \phi_g\left((r+\delta) - \rho + \psi_b\right)}{2(r+\delta)\left(r+\delta-\rho+\psi_b\right)} + \frac{\rho S_\infty\left(-2\psi_b^2 + 2\gamma\psi_g\psi_b + \psi_g - 2\right)}{2(r+\delta)\left(r+\delta-\rho+\psi_b\right)}$$

and

$$\tilde{n}_b(\phi_b,\, \psi_b,\, \phi_g,\, \psi_g) = -\frac{\psi_g + 2\psi_b\left(r + \delta - \rho + \gamma\psi_g\right) - 2}{2\left(r + \delta - \rho + \psi_b\right)}$$

Finally, Plugging $(\hat{\phi}_g, \hat{\psi}_g, \hat{\phi}_b, \hat{\psi}_b)$ back into the expressions for $\hat{m}_b$ and $\hat{n}_b$, and using the fact that $\rho^2 = r\rho + \delta(r+\delta) + 1/(\gamma^2 - 1)$ to rearrange terms, we obtain Equations (68-69).

B.2 Populist policy (general)

We follow the same steps as in the previous section. Using (44) to eliminate $\lambda_a(t)$ from Equation (46), we obtain the following conditon :

$$\dot{S}\,\left(1/2 + \Phi_b'(S) + \gamma\,\Phi_g'(S)\right) - (r+\delta)\left(-a + S/2 + \Phi_b(S) + \gamma\,\Phi_g(S)\right) + (1/2)\left(\Phi_g(S) - \Phi_b(S)\right) - \left(1 - \frac{1}{6\,(1-\gamma)}\right)\,S = 0 \quad (88)$$

Now, substracting Condition (86) from Condition (88), we get:

$$\dot{S}\,\left(\tau_b'(S) + \Phi_b'(S)\right) - (r+\delta)\left(\tau_b(S) + \Phi_b(S)\right) - \left(\gamma\,\Phi_g'(S) + \tau_b'(S)\right)\Phi_b(S) - \frac{1}{2}\,\Phi_g(S) + \left(1 - \frac{1}{6\,(1-\gamma)}\right)\,S = 0 \quad (89)$$

Following the same steps as above, condition (88) can be rewritten as $\alpha_p(m_b,\, n_b) + \beta_p(m_b,\, n_b)\,(\bar{S} - S_p^{\infty})e^{\rho t} = 0$ where

$$\alpha_p(m_b,\, n_b) = -(r+\delta)m_b - (1/2)\left(\phi_g + 2\phi_b\left(r + \delta + n_b + \gamma\psi_g\right)\right) + \left(\beta_p(m_b,\, n_b) - \rho(n_b + \psi_b)\right)\,S_p^{\infty}$$

$$\beta_p(m_b,\, n_b) = \frac{6\gamma - 5}{6\,(\gamma - 1)} - \left(n_b\left(r + \delta - \rho + \psi_b\right) + (1/2)\psi_g + \psi_b\left(r + \delta - \rho + \gamma\psi_g\right)\right)$$

Solving the system$\{\alpha_o(m_b, n_b) = 0,\ \beta_o(m_b, n_b) = 0\}$ for m_b and n_b, we get:

$$\check{m}_b(\phi_b,\, \psi_b,\, \phi_g,\, \psi_g) = -\frac{\left(6(\gamma-1)\psi_b^2 + \gamma - 3(\gamma-1)\left(2\gamma\psi_b + 1\right)\psi_g - 5\right)\rho S_\infty}{6(\gamma-1)(r+\delta)\left(r + \delta - \rho + \psi_b\right)} - \frac{3\,\phi_g\left(r + \delta - \rho + \psi_b\right)}{6(r+\delta)\left(r + \delta - \rho + \psi_b\right)} - \frac{\phi_b\left(6\gamma + 6(\gamma-1)(r+\delta)(r+\delta-\rho) + 6(\gamma-1)\rho\psi_b + 3(\gamma-1)(2\gamma(r+\delta-\rho) - 1)\psi_g - 5\right)}{6(\gamma-1)(r+\delta)\left(r + \delta - \rho + \psi_b\right)} \quad (90)$$

and

$$\check{n}_b(\phi_b,\, \psi_b,\, \phi_g,\, \psi_g) = -\frac{-6\gamma + 3(\gamma-1)\psi_g + 6(\gamma-1)\psi_b\left(r + \delta - \rho + \gamma\psi_g\right) + 5}{6(\gamma-1)\left(r + \delta - \rho + \psi_b\right)}$$

Finally, Plugging $(\check{\phi}_g, \check{\phi}_g, \check{\phi}_b, \check{\phi}_b)$ back into the expressions for $\check{m}_b$ and $\check{n}_b$, and using the fact that $\rho^2 = r\rho + \delta(r+\delta) + 1/(\gamma^2 - 1)$ to rearrange terms, we obtain Equations (79-80).